Unitary correlation sets and their applications

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter §2 is based on a paper by the author [31], which has appeared in the Indiana University Mathematics Journal. Chapter §3 arises from joint work with Vern Paulsen [32] in Integral Equations and Operator Theory. Chapter §4 is from a paper by the author that has appeared in the Journal of Mathematical Physics [29]. Chapter §5 is based on a paper co-authored with Li Gao and Marius Junge [26], which has appeared in the International Mathematics Research Notices. Chapter §6 comes from a preprint by the author [30].

Abstract

We relate Connes' embedding problem in operator algebras to the Brown algebra $\mathcal{U}_{nc}(n)$, which is defined as the universal unital C^*-algebra generated by the entries of an $n \times n$ unitary matrix. In particular, we show that the embedding problem is equivalent to determining whether or not the algebra $\mathcal{U}_{nc}(n)$ has the weak expectation property for any (equivalently, all) $n \geq 2$. From this perspective, we develop a theory of what we call unitary correlation sets. These sets are analogous to the usual sets of bipartite probabilistic correlations arising from the typical models in quantum information theory. We show that the analogue of the weak Tsirelson problem for unitary correlation sets is again equivalent to Connes' embedding problem. Moreover, we show that as long as Alice and Bob's unitaries are of size at least 2, the set of spatial unitary correlations is never a closed set. This result is analogous to a recent theorem of Slofstra, which states that the set of quantum probabilistic correlations is not closed, so long as the input and output sets are large enough.

We also discuss several applications of the theory of unitary correlation sets. First, we show that the class of extended non-local games known as quantum XOR games is a rich enough class to detect the validity of Connes' embedding problem. That is, the embedding problem has a positive answer if and only if the value of every quantum XOR game in the commuting model agrees with the value of the game in the approximate finite-dimensional model. Second, we use a C^*-algebraic analogue of the quantum teleportation and super-dense coding maps from quantum information theory to obtain separations between the tensor product model (or "quantum spatial" model) and the approximate finite-dimensional model (or "quantum approximate" model), for matrix-valued generalizations of the usual Tsirelson corrleation sets. We use some of the intermediate results to also obtain separations between the matrix versions of the finite-dimensional model (or "quantum model") and the tensor product model.

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Dedication

To Jenn, Abby, Naomi, and (soon to arrive) Baby Harris.

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Chapter 1

Introduction

The content of this thesis lies on the bridge between quantum information theory and the theory of operator algebras. In particular, we consider some new avenues for two equivalent open problems: Connes' embedding problem (in operator algebras), and Tsirelson's problem (in quantum information theory). Connes' embedding problem is perhaps the most important open problem in operator algebras. It asks whether every weakly separable, finite von Neumann algebra can be approximately embedded into the unique hyperfinite II_1 factor von Neumann algebra in a trace-preserving manner [13]. While this problem appears, at first, to be a very technical question, it turns out to be equivalent to many other significant open problems in operator algebras, and even some open problems in entirely different areas of mathematics. For example, E. Kirchberg proved that Connes' embedding problem has a positive answer if and only if the full group C^*-algebra of the free group on (countably) infinitely many generators has a unique C^*-norm when tensored with itself [41]. In the language of free entropy, D. Voiculescu proved that the embedding problem is equivalent to the existence of microstates [66]. Connes' embedding problem is also equivalent to a non-commutative version of Hilbert's 17th problem [55], and has equivalent statements in terms of sums of squares problems (see, for example, [36, 43]), which arise in non-commutative algebraic geometry.

One of the more recent restatements of Connes' embedding problem is Tsirelson's problem in quantum information theory [25, 34, 47]. This problem concerns bipartite separated systems. If Alice and Bob have labs in a finite-input, finite-output separated system, then the probability that Alice and Bob obtain certain outputs after measurements, given that they started with certain inputs, can be modelled by quantum mechanics. Depending on the model used, the set of possible probabilistic correlations changes. The most commonly used correlation sets correspond to the finite-dimensional tensor product model, the tensor product model, the approximate finite-dimensional model, and the commuting model, respectively. For m inputs and k outputs, the probabilistic correlation sets in these models are typically denoted by $C_q(m,k)$, $C_{qs}(m,k)$, $C_{qa}(m,k)$ and $C_{qc}(m,k)$, respectively. It is well-known (see, for example, [25, 34, 47]) that

$$C_q(m,k) \subseteq C_{qs}(m,k) \subseteq C_{qa}(m,k) \subseteq C_{qc}(m,k) \subseteq \mathbb{R}^{m^2k^2},$$

and that $C_{qa}(m,k)$ is equal to the closure of $C_{qs}(m,k)$ and the closure of $C_q(m,k)$. Tsirelson's problem asks whether $C_{qa}(m,k) = C_{qc}(m,k)$ for all $m, k \geq 2$ [62]. In particular, the problem asks whether every probability distribution in this setting in a commuting operator framework can be approximated arbitrarily well by probability distributions in the finite-dimensional tensor product setting. The commuting operator framework comes from the axioms of quantum field theory, whereas the finite-dimensional tensor product comes from the natural assumptions of quantum mechanics. Hence, determining the solution to Connes' embedding problem would be a significant advancement in several areas of mathematics.

Related to Tsirelson's problem is determining whether either of $C_q(m,k)$ or $C_{qs}(m,k)$ is a closed set for all m, k. This answer was only recently resolved in the negative by W. Slofstra [60]. Before Slofstra's work, it was shown by R. Cleve, L. Liu and V. Paulsen that embezzling entanglement is a protocol that can be done perfectly in a commuting operator framework (and even in an approximate finite-dimensional operator framework), but not in a tensor product framework [11]. More specifically, they showed that, if Alice and Bob have resource Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, so that $\mathcal{H}_A \otimes \mathcal{H}_B$ is initialized in the state ψ, and if each of Alice and Bob have a state space $\mathbb{C}^2$, then there do not exist local unitaries $U \in \mathcal{B}(\mathbb{C}^2 \otimes \mathcal{H}_A)$ and $V \in \mathcal{B}(\mathcal{H}_B \otimes \mathbb{C}^2)$ such that

$$(U \otimes V)(e_0 \otimes \psi \otimes e_0) = \frac{1}{\sqrt{2}}(e_0 \otimes \psi \otimes e_0 + e_1 \otimes \psi \otimes e_1).$$

This process, if it is possible, is referred to as **embezzlement of entanglement**. In contrast, if one replaces their resource Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ with a infinite-dimensional Hilbert space $\mathcal{H}$ and assumes that Alice and Bob are in a commuting operator framework (rather than a tensor product framework), then embezzling entanglement is possible [11]. The fact that embezzlement of entanglement can always be approximately achieved in (large enough) finite dimensions was shown by W. van Dam and P. Hayden [64].

This thesis concerns an operator algebraic approach to Connes' embedding problem where the correlation sets obtained can precisely describe embezzlement of entanglement. The C^*-algebra in the background of much of this work is the Brown algebra, which is the universal C^*-algebra $\mathcal{U}_{nc}(n)$ generated by the n^2 entries of an $n \times n$ unitary matrix. This C^*-algebra was defined by L. Brown [7]. In Chapter 2, we obtain an equivalent statement of Kirchberg's conjecture (and hence Connes' embedding problem) in terms of $\mathcal{U}_{nc}(n)$. We also show that the Brown operator system $\mathcal{V}_n$ spanned by the generators of $\mathcal{U}_{nc}(n)$ arises as an operator system quotient of the space M_{2n} of $2n \times 2n$ matrices. These facts lay the groundwork for developing a theory of what we call **unitary correlation sets** in Chapter 3. These correlations are similar to the probabilistic correlation sets in Tsirelson's problem, except that unitary correlations can be used to encode embezzlement of entanglement. Similar to the probabilistic correlations above, we obtain unitary correlation sets $B_q(n,m)$, $B_{qs}(n,m)$, $B_{qa}(n,m)$, and $B_{qc}(n,m)$, where n and m now represent the matrix size of Alice and Bob's unitaries, respectively. Moreover, we formulate a problem regarding the unitary correlation sets that is both analogous and equivalent to the original Tsirelson's problem (and hence equivalent to Connes' embedding problem). That is to say, Connes' embedding problem is equivalent to determining whether $B_{qa}(n,m) = B_{qc}(n,m)$ for all $n,m \geq 2$. The closure question for unitary correlations has a negative answer: neither $B_q(n,m)$ nor $B_{qs}(n,m)$ is closed if $n,m \geq 2$. The proof of this fact is quite elementary and relies on embezzlement of entanglement.

After developing the theory of unitary correlation sets, we discuss two main applications of these sets. The first application is in Chapter 4, where we show that the unitary correlation sets can be thought of as the sets of possible strategies for a class of two-player, extended non-local games known as quantum XOR games, which were introduced by O. Regev and T. Vidick [54]. Moreover, we prove that Connes' embedding problem is equivalent to determining whether, for every quantum XOR game, the optimal winning probability in the commuting model is the same as the optimal winning probability in the approximate finite-dimensional model. In other words, the class of quantum XOR games is rich enough to detect whether or not Connes' embedding problem has a positive answer. Whether a class of two-player non-local games (with classical questions rather than quantum questions) exists with this property is not yet known.

Our second application of unitary correlations is in Chapter 5. It comes from a C^*-algebraic analogue of two fundamental protocols in quantum information theory: quantum teleportation and super-dense coding. These protocols have already been interpreted at the level of operator spaces in [35], using the trace-class operators S_1^d on d-dimensional space and the d-dimensional sequence space ℓ_1^d, respectively. We demonstrate that the correct C^*-algebras to use for a C^*-algebraic analogue are the universal C^*-algebra $\mathcal{U}_{nc}(d)$ generated by the entries of a $d \times d$ unitary matrix (sometimes referred to as the Brown algebra), and the full group C^*-algebra $C^*(\mathbb{F}_{d^2})$ of the free group on d^2 generators. Using certain group actions of $\mathbb{Z}_d$ on the algebras $\mathcal{U}_{nc}(d)$ and $C^*(\mathbb{F}_{d^2})$ respectively, we prove that

$$\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \simeq M_d(C^*(\mathbb{F}_{d^2})) \text{ and } C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d \simeq M_d(\mathcal{U}_{nc}(d)).$$

These isomorphisms are explicit. Using the first isomorphism, we obtain separations of the approximate finite-dimensional model from the tensor product model for a matrix-valued generalization of the probabilistic correlation sets. In particular, if $t \in \{q, qs, qa, qc\}$ and $C_t^{(n)}(m,k)$ denotes the matrix-valued generalization of $C_t(m,k)$ where the probability density is replaced by an M_n-valued probability density, then $C_{qs}^{(5)}(3,2) \neq C_{qa}^{(5)}(3,2)$ and $C_{qs}^{(13)}(2,3) \neq C_{qs}^{(13)}(2,3)$. Since the smallest known separation in the $n=1$ case is that $C_{qs}(5,2) \neq C_{qa}(5,2)$ [20], we have separated the two models for the matrix-valued correlations in situations where the analogous separation for probabilistic correlations is not yet known. Finally, we use some of the techniques from Chapter 5 to obtain separations between the finite-dimensional and the tensor product models for matrix-valued correlations in Chapter 6. In particular, we prove that $C_q^{(3)}(3,2) \neq C_{qs}^{(3)}(3,2)$ and $C_q^{(4)}(2,3) \neq C_{qs}^{(4)}(2,3)$. In comparison, the smallest known values of m and k for which $C_q(m,k) \neq C_{qs}(m,k)$ are $m = 5$ and $k = 3$ [12].

This thesis is roughly the compilation of the papers [31], [32], [29], [26], and [30], which correspond to Chapters 2, 3, 4, 5 and 6, respectively. In the rest of Chapter 1, we introduce the necessary background theory that we need. We cover operator systems, their duals, and their quotients in Section 1.1. In Section 1.2, we will recall the basics of tensor products of Banach spaces, Hilbert spaces, C^*-algebras and operator systems. Section 1.3 covers certain finite-dimensional approximation properties of C^*-algebras and operator systems that are relevant to Connes' embedding problem. In Section 1.4, we will demonstrate some group embeddings that factor into our separations of matrix-valued correlation sets in Chapters 5 and 6. We recall the basic theory of crossed products in Section 1.5. Section 1.6 treats the definitions for the free product of unital C^*-algebras amalgamated over the identity, along with the reduced free product. Finally, we consider the Tsirelson problems and their connection with Connes' embedding problem in Sections 1.7–1.9.

Throughout the thesis, we will assume a general familiarity with functional analysis and the theory of C^*-algebras. We refer the reader to [14] for a thorough introduction to functional analysis. For C^*-algebras, we refer the reader to [8] and [15].

1.1 Operator systems

In this section, we will briefly give some background on operator systems that we will use throughout the thesis. In particular, we will give the abstract definition of operator systems, which will allow us to consider operator system duals and operator system quotients.

1.1.1 Abstract characterization of operator systems

Here we define abstract operator systems. More information on abstract operator systems can be found in [49, Chapter 13]. The abstract definition of operator systems has many advantages. In particular, there is no need to specify the ambient Hilbert space on which an operator system acts, and one can consider an operator system purely in terms of its involution, positive cones, and order unit. This link is due to Theorem 1.1.1, which is a celebrated theorem of Choi and Effros [10].

We begin with some definitions. A (complex) vector space $\mathcal{S}$ is called a **$*$-vector space** if it is equipped with a mapping $* : \mathcal{S} \to \mathcal{S}$ such that, for all $a, b \in \mathcal{S}$ and $\lambda \in \mathbb{C}$,

- $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$; and
- $(a^*)^* = a$.

The map $*$ is often called an **involution**, and we typically refer to x^* as the **adjoint** of x, where $x \in \mathcal{S}$. An element $x \in \mathcal{S}$ is **self-adjoint** (or **hermitian**) if $x = x^*$. We set

$$\mathcal{S}_h = \{x \in \mathcal{S} : x = x^*\},$$

which is a real vector space. If $\mathcal{S}$ is a $*$-vector space, then the space $M_n(\mathcal{S})$ of $n \times n$ matrices with entries in $\mathcal{S}$ has a natural $*$-vector space structure given by $(x_{ij})^* = (x_{ji}^*)$. A **matrix ordering** on a $*$-vector space $\mathcal{S}$ is a collection of subsets $(C_n)_{n=1}^\infty$, where $C_n \subseteq (M_n(\mathcal{S}))_h$, such that, for all $n, m \in \mathbb{N}$,

- $C_n + C_n \subseteq C_n$ and $tC_n \subseteq C_n$ for each $t \geq 0$;
- $C_n \cap (-C_n) = \{0\}$; and
- $AXA^* \in C_m$ whenever $X \in C_n$ and $A \in M_{m,n}(\mathbb{C})$.

If $\mathcal{S}$ is a $*$-vector space with matrix ordering $(C_n)_{n=1}^\infty$, then we call $(\mathcal{S}, (C_n)_{n=1}^\infty)$ a **matrix-ordered $*$-vector space**. We often refer to C_n as the cone of positive elements of $M_n(\mathcal{S})$, and write $M_n(\mathcal{S})_+ = C_n$ and $X \geq 0$ if $X \in M_n(\mathcal{S})_+$. An **order unit** for $\mathcal{S}$ is an element $e \in \mathcal{S}_h$ such that, for any $x \in \mathcal{S}_h$, there is $r > 0$ such that $rx + e \in C_1$. We call e a **matrix**

order unit if, for each $n \in \mathbb{N}$, the $n \times n$ diagonal matrix $I_n := \begin{pmatrix} e & & \\ & \ddots & \\ & & e \end{pmatrix}$ is an order unit for $M_n(\mathcal{S})$ with respect to C_n. An order unit e for $\mathcal{S}$ is called **Archimedean** if, whenever $x \in \mathcal{S}_h$ is such that $x + re \in C_1$ for all $r > 0$, then $x \in C_1$. If I_n is an Archimedean order unit for $M_n(\mathcal{S})$ with respect to C_n for each $n \in \mathbb{N}$, then e is called an **Archimedean matrix order unit**. Lastly, if $\mathcal{S}$ is a matrix-ordered $*$-vector space with matrix ordering $(C_n)_{n=1}^{\infty}$ and Archimedean matrix order unit e, then we call the triple $(\mathcal{S}, (C_n)_{n=1}^{\infty}, e)$ an **(abstract) operator system**.

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. For a linear map $\varphi : \mathcal{S} \to \mathcal{T}$, we let $\varphi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ be the linear map given by $\varphi^{(n)}((x_{ij})) = (\varphi(x_{ij}))$. A linear map $\varphi : \mathcal{S} \to \mathcal{T}$ is **positive** if $\varphi(\mathcal{S}_+) \subseteq \mathcal{T}_+$. If $n \in \mathbb{N}$, then we say that φ is n**-positive** if $\varphi^{(n)}$ is positive. We say that φ is **completely positive** if it is n-positive for all $n \in \mathbb{N}$. We will often write the abbreviation "cp" for "completely positive", and "ucp" for "unital and completely positive". Note that if the map φ is n-positive, then it is automatically k-positive for all $k \leq n$. We call the map φ an **order isomorphism** if it is positive, a bijection, and its inverse $\varphi^{-1} : \mathcal{T} \to \mathcal{S}$ is positive. We call φ a **complete order isomorphism** if φ and φ^{-1} are completely positive. Finally, if $\varphi : \mathcal{S} \to \mathcal{T}$ is a linear map and $\mathcal{R}$ is the range of φ, we call φ a **complete order embedding** if φ is a complete order isomorphism onto $\mathcal{R}$, where $\mathcal{R}$ is equipped with the involution from $\mathcal{T}$ and positive cones $D_n = C_n \cap \mathcal{R}$ for all $n \in \mathbb{N}$.

Thanks to the following theorem, each operator system can be regarded as a self-adjoint, unital subspaces of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 1.1.1. (Choi-Effros, [10]) *Let $(\mathcal{S}, (C_n)_{n=1}^{\infty}, e)$ be an abstract operator system. Then there is a Hilbert space $\mathcal{H}$ and a complete order embedding $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ with $\varphi(e) = I_{\mathcal{H}}$. Conversely, if $\mathcal{T}$ is a self-adjoint subspace of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$, then $\mathcal{T}$ is an operator system with Archimedean matrix order unit $I_{\mathcal{H}}$.*

The operator system $\mathcal{B}(\mathcal{H})$ possesses two relevant properties. First, $\mathcal{B}(\mathcal{H})$ is a unital C^*-algebra, which implies that any operator system can be embedded into a unital C^*-algebra. Second, by Arveson's extension theorem [2], $\mathcal{B}(\mathcal{H})$ is injective in the category of operator systems (with morphisms given by ucp maps). That is to say, whenever $\mathcal{S}$ and $\mathcal{T}$ are operator systems with $\mathcal{S} \subseteq \mathcal{T}$ and $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is a ucp map, then there is a ucp map $\widetilde{\varphi} : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ such that $\widetilde{\varphi}_{|\mathcal{S}} = \varphi$. In particular, an operator system $\mathcal{S}$ can always be embedded into some injective C^*-algebra. Hence, one may consider the smallest C^*-algebra generated by a (unital completely order isomorphic) copy of an operator system $\mathcal{S}$.

Similarly, one may consider the smallest injective operator system that contains $\mathcal{S}$. These two objects are known as the C^*-envelope and the injective envelope, respectively.

If $\mathcal{S}$ is an operator system, then a C^*-**envelope** of $\mathcal{S}$ is a pair $(C_e^*(\mathcal{S}), \iota)$, where $C_e^*(\mathcal{S})$ is a C^*-algebra and $\iota : \mathcal{S} \to C_e^*(\mathcal{S})$ is a unital complete order embedding such that $C_e^*(\mathcal{S}) = C^*(\iota(\mathcal{S}))$, with the following property:

- If $\mathcal{A}$ is another C^*-algebra and $\kappa : \mathcal{S} \to \mathcal{A}$ is a unital complete order embedding with $C^*(\kappa(\mathcal{S})) = \mathcal{A}$, then there is a unique, surjective $*$-homomorphism $\pi_e : \mathcal{A} \to C_e^*(\mathcal{S})$ such that $\pi_e(\kappa(s)) = \iota(s)$ for all $s \in \mathcal{S}$.

Evidently, the C^*-envelope plays the role of the smallest possible C^*-cover for the operator system $\mathcal{S}$. One can also define an **injective envelope** of $\mathcal{S}$, as a pair $(\mathcal{I}(\mathcal{S}), \kappa)$, where $\kappa : \mathcal{S} \to \mathcal{I}(\mathcal{S})$ is a unital complete order embedding, $\mathcal{I}(\mathcal{S})$ is an injective operator system, and whenever $\mathcal{T}$ is an injective operator system with $\kappa(\mathcal{S}) \subseteq \mathcal{T} \subseteq \mathcal{I}(\mathcal{S})$, then $\mathcal{T} = \mathcal{I}(\mathcal{S})$ (see, for example, [49, Chapter 15]). M. Hamana proved [27] that the C^*-envelope and the injective envelope of an operator system always exist, and that these two objects are unique up to an isomorphism that fixes the copy of $\mathcal{S}$. For this reason, we will often refer to "the" C^*-envelope of an operator system $\mathcal{S}$ and "the" injective envelope of $\mathcal{S}$.

1.1.2 Operator system dual

In this subsection, we describe the positivity structure on dual spaces of operator systems. In general, a dual space of an operator system need not be an operator system, since there may be no order unit. However, in finite dimensions, an order unit will always exist, and this fact will be helpful for our purposes.

While operator systems are often studied with respect to their positivity structure, there is also an associated norm structure on an operator system $\mathcal{S}$ (see, for example, [49, Chapter 13]). Indeed, if $(\mathcal{S}, (C_n)_{n=1}^\infty, e)$ is an abstract operator system and $X \in M_n(\mathcal{S})$, then the norm of X is given by

$$\|X\|_{M_n(\mathcal{S})} = \inf \left\{ r > 0 : \begin{pmatrix} rI_n & X \\ X^* & rI_n \end{pmatrix} \in C_{2n} \right\}.$$

Consider an operator system $\mathcal{S}$ with Banach space dual $\mathcal{S}^d$. There is a natural way to give $\mathcal{S}^d$ the structure of a matrix-ordered $*$-vector space. Indeed, for a functional $\varphi \in \mathcal{S}^d$, we define $\varphi^*(s) = \overline{\varphi(s^*)}$. Similarly, given $f = (f_{ij}) \in M_n(\mathcal{S}^d)$, we define $f^* = (f_{ji}^*)$. We say that an element $f = (f_{ij}) \in M_n(\mathcal{S}^d)$ is positive provided that the associated linear map

$F : \mathcal{S} \to M_n$ given by $F(s) = (f_{ij}(s))$ is completely positive. With these definitions, $\mathcal{S}^d$ has the structure of a matrix-ordered $*$-vector space [10]. In general, $\mathcal{S}^d$ may not have an order unit. However, when $\mathcal{S}$ is finite-dimensional, then there exists a faithful state φ on $\mathcal{S}$; that is, there is a state φ such that $\varphi(s) > 0$ for all $s \in \mathcal{S}_+ \setminus \{0\}$. Then any faithful state on $\mathcal{S}$ will be an Archimedean matrix order unit for $\mathcal{S}$ [10]. In this way, $\mathcal{S}^d$ becomes an operator system when $\mathcal{S}$ is finite-dimensional. Moreover, in the finite-dimensional case, the canonical bijective map $\iota : \mathcal{S} \to \mathcal{S}^{dd}$ given by $[\iota(s)](f) = f(s)$ is a unital complete order isomorphism.

One of the most important examples of an operator system dual is M_n. Indeed, this operator system is self-dual in the following sense:

Theorem 1.1.2. (Paulsen-Todorov-Tomforde, [52, Theorem 6.2]) *Let $\{E_{ij}\}_{i,j=1}^n$ be the standard matrix units for M_n, and let $\{\delta_{ij}\}_{i,j=1}^n$ be the basis of dual functionals for M_n^d given by*

$$\delta_{ij}(E_{k\ell}) = \begin{cases} 1 & i = k \text{ and } j = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then the map $\delta_{ij} \mapsto E_{ij}$ extends to a complete order isomorphism $M_n^d \simeq M_n$. Moreover, equipping M_n^d with the order unit given by the unnormalized trace functional $Tr = \sum_{i=1}^n \delta_{ii}$, the map is a unital complete order isomorphism.

Theorem 1.1.2 shows the difference between the structure of the operator system dual and the operator space dual. Indeed, with respect to the operator space dual structure of M_n, the trace functional Tr has norm n, even though the trace functional is the order unit in M_n^d (and hence has norm 1 with respect to the operator system dual structure of M_n^d).

1.1.3 Quotients

The theory of operator system quotients is still very new in the literature. In this subsection we outline some of the theory of operator system quotients. More information can be found in [39]. Given an operator system $\mathcal{S}$ and a self-adjoint subspace $\mathcal{J} \subseteq \mathcal{S}$ with $1_{\mathcal{S}} \notin \mathcal{J}$, we call $\mathcal{J}$ a **kernel** provided that there is an operator system $\mathcal{T}$ and a ucp map $\varphi : \mathcal{S} \to \mathcal{T}$ such that $\ker(\varphi) = \mathcal{J}$. Given a kernel $\mathcal{J}$ of an operator system $\mathcal{S}$, we may endow the quotient vector space $\mathcal{S}/\mathcal{J}$ with an operator system structure as follows. Let $q : \mathcal{S} \to \mathcal{S}/\mathcal{J}$ be the canonical (vector space) quotient map. We define an involution on $\mathcal{S}/\mathcal{J}$ by setting $(q(x))^* = q(x^*)$ for all $x \in \mathcal{S}$. For $n \in \mathbb{N}$, we define

$$\mathcal{D}_n(\mathcal{S}, \mathcal{J}) = \{X \in M_n(\mathcal{S}/\mathcal{J})_h : X = q^{(n)}(Y) \text{ for some } Y \in M_n(\mathcal{S})_+\}.$$

In general, these cones may not satisfy the Archimedean property with respect to the order unit $1_{\mathcal{S}/\mathcal{J}} = q(1_{\mathcal{S}})$ (see [39]). If we define

$$\mathcal{C}_n(\mathcal{S},\mathcal{J}) = \{X \in M_n(\mathcal{S}/\mathcal{J})_h : X + \varepsilon I_n \in \mathcal{D}_n(\mathcal{S},\mathcal{J}) \text{ for all } \varepsilon > 0\},$$

Then $(\mathcal{S}/\mathcal{J}, (\mathcal{C}_n)_{n=1}^{\infty}, 1_{\mathcal{S}/\mathcal{J}})$ is an operator system [39].

We call a kernel $\mathcal{J}$ of an operator system $\mathcal{S}$ **completely order proximinal** if, for each $n \in \mathbb{N}$, we have $\mathcal{D}_n(\mathcal{S},\mathcal{J}) = \mathcal{C}_n(\mathcal{S},\mathcal{J})$. In the finite-dimensional setting, there is a very useful criterion for finding (completely order proximinal) kernels.

Proposition 1.1.3. (Kavruk, [37, Proposition 2.4]) *Let $\mathcal{S}$ be a finite-dimensional operator system. If $\mathcal{J}$ is a self-adjoint subspace of $\mathcal{S}$ with no positive or negative elements except 0, then $\mathcal{J}$ is the kernel of a ucp map on $\mathcal{S}$. Moreover, $\mathcal{J}$ is completely order proximinal.*

In general, the first isomorphism theorem fails for surjective ucp maps between operator systems. In other words, a surjective ucp map need not induce a complete order isomorphism on the quotient operator system. Nevertheless, the following result allows us to translate surjective ucp maps to ucp maps on the quotient system.

Proposition 1.1.4. (Kavruk-Paulsen-Todorov-Tomforde, [39, Proposition 3.6]) *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems, and let $\mathcal{J}$ be a kernel in $\mathcal{S}$. If $\varphi : \mathcal{S} \to \mathcal{T}$ is ucp and $\mathcal{J} \subseteq \ker(\varphi)$, then the map $\dot{\varphi} : \mathcal{S}/\mathcal{J} \to \mathcal{T}$ given by $\dot{\varphi}(x + \mathcal{J}) = \varphi(x)$ is well-defined and ucp.*

Given operator systems $\mathcal{S}$ and $\mathcal{T}$ and a surjective ucp map $\varphi : \mathcal{S} \to \mathcal{T}$, we say that φ is a **complete quotient map** provided that the induced map $\dot{\varphi} : \mathcal{S}/\ker(\varphi) \to \mathcal{T}$ from Proposition 1.1.4 is a complete order isomorphism. In the finite-dimensional setting, there is a direct relation between complete quotient maps and complete order embeddings. Indeed, if $\varphi : \mathcal{S} \to \mathcal{T}$ is a ucp map, then the adjoint map $\varphi^d : \mathcal{T}^d \to \mathcal{S}^d$ is given by $[\varphi^d(\psi)](s) = \psi(\varphi(s))$ for all $\psi \in \mathcal{T}^d$ and $s \in \mathcal{S}$. The following proposition links complete quotient maps and complete order embeddings.

Proposition 1.1.5. (Farenick-Paulsen, [24, Proposition 1.8]) *Let $\mathcal{S}$ and $\mathcal{T}$ be finite-dimensional operator systems. Then a linear map $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete quotient map if and only if the adjoint map $\varphi^d : \mathcal{T}^d \to \mathcal{S}^d$ given by $[\varphi^d(\psi)](s) = \psi(\varphi(s))$ is a complete order embedding.*

1.2 Tensor Products

Throughout the thesis, we require different kinds of tensor products of spaces. In the subsections that follow, we will outline the definitions and facts of interest to us regarding tensor products of Banach spaces, Hilbert spaces, C^*-algebras, and operator systems.

1.2.1 Tensor products of Banach spaces

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A **reasonable cross-norm** on the vector space $\mathcal{X} \otimes \mathcal{Y}$ is a norm $\alpha : \mathcal{X} \otimes \mathcal{Y} \to \mathbb{C}$ such that

- For all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $\alpha(x \otimes y) \leq \|x\|_{\mathcal{X}} \|y\|_{\mathcal{Y}}$; and
- For all $\varphi \in \mathcal{X}^*$ and $\psi \in \mathcal{Y}^*$, the functional $\varphi \otimes \psi : \mathcal{X} \otimes \mathcal{Y} \to \mathbb{C}$ is bounded with respect to α, with norm at most $\|\varphi\|_{\mathcal{X}^*} \|\psi\|_{\mathcal{Y}^*}$.

There are many examples of reasonable cross-norms. There is a largest reasonable cross-norm and a smallest reasonable cross-norm; these are called the projective (Banach space) tensor norm and the injective (Banach space) tensor norm, respectively.

Definition 1.2.1. *The* ***projective (Banach space) tensor norm*** *on $\mathcal{X} \otimes \mathcal{Y}$ is defined as*

$$\|z\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : x_i \in \mathcal{X},\, y_i \in \mathcal{Y},\, z = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

The ***projective (Banach space) tensor product*** *of $\mathcal{X}$ and $\mathcal{Y}$, denoted by $\mathcal{X} \widehat{\otimes}_\pi \mathcal{Y}$, is the completion of $\mathcal{X} \otimes \mathcal{Y}$ with respect to $\| \cdot \|_\pi$. (We often denote by $\mathcal{X} \otimes_\pi \mathcal{Y}$ the vector space $\mathcal{X} \otimes \mathcal{Y}$ equipped with the norm $\| \cdot \|_\pi$.)*

There are many equivalent descriptions of the projective Banach space tensor norm. The following is a helpful description of the unit ball in the projective Banach space tensor product:

Proposition 1.2.2. (See [56, Proposition 2.2]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. Then the unit ball of $\mathcal{X} \widehat{\otimes}_\pi \mathcal{Y}$ is given by the closed convex hull of the set of all elements of the form $x \otimes y$, where $x \in \mathcal{X}$ with $\|x\| \leq 1$ and $y \in \mathcal{Y}$ with $\|y\| \leq 1$.*

An application of Proposition 1.2.2 shows that an element $u \in \mathcal{X} \widehat{\otimes}_\pi \mathcal{Y}$ satisfies $\|u\|_\pi < 1$ if and only if we can write

$$u = \sum_{n=1}^{\infty} x_n \otimes y_n,$$

where $x_n \in \mathcal{X}$ and $y_n \in \mathcal{Y}$ satisfy

$$\sum_{n=1}^{\infty} \|x_n\| \|y_n\| < 1.$$

It is also helpful to consider the projective Banach space tensor product from the perspective of bounded bilinear mappings. Given two Banach spaces $\mathcal{X}$ and $\mathcal{Y}$, a bilinear form $B : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$ is said to be bounded if there is $C > 0$ such that $|B(x, y)| \leq C\|x\|\|y\|$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For a bounded bilinear form B, we let

$$\|B\| = \inf\{C > 0 : |B(x, y)| \leq C\|x\|\|y\| \text{ for all } x \in \mathcal{X} \text{ and } y \in \mathcal{Y}\}.$$

Then $\|\cdot\|$ defines a norm on the vector space $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$ of bounded bilinear forms on $\mathcal{X} \times \mathcal{Y}$; moreover, $\mathcal{B}(\mathcal{X} \times \mathcal{Y})$ is complete with respect to this norm. The perspective of bounded bilinear forms yields the following description of the projective norm:

Theorem 1.2.3. (See [56, p. 23]) *If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces and $z \in \mathcal{X} \otimes \mathcal{Y}$, then*

$$\|z\|_\pi = \sup\{|B(z)| : B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y}),\ \|B\| \leq 1\}.$$

Let $B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$ be a bounded bilinear form. Using Theorem 1.2.3, we see that the linearization $\widetilde{B}$ of B extends to a continuous linear functional on $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$; moreover, $\|\widetilde{B}\|_{\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}} = \|B\|$. Conversely, if $u \in (\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^*$, then there is an associated bounded bilinear form $B_u \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$ such that $B_u(x, y) = u(x \otimes y)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$; moreover, $\|B_u\| = \|u\|_{(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^*}$. In particular, there is an isometric isomorphism $(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^* \simeq \mathcal{B}(\mathcal{X}\times\mathcal{Y})$. Considering this identification and the dual of the projective tensor product naturally leads to the following definition.

Definition 1.2.4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. The* ***injective (Banach space) tensor norm*** *on $\mathcal{X} \otimes \mathcal{Y}$ is defined as*

$$\|z\|_\varepsilon = \sup\{|(\varphi \otimes \psi)(z)| : \varphi \in \mathcal{X}^*,\ \psi \in \mathcal{Y}^*,\ \|\varphi\| \leq 1,\ \|\psi\| \leq 1\}.$$

The ***injective (Banach space) tensor product*** *of $\mathcal{X}$ and $\mathcal{Y}$, denoted by $\mathcal{X}\widehat{\otimes}_\varepsilon\mathcal{Y}$, is the completion of $\mathcal{X} \otimes \mathcal{Y}$ with respect to the norm $\|\cdot\|_\varepsilon$. (We often denote by $\mathcal{X} \otimes_\varepsilon \mathcal{Y}$ the vector space $\mathcal{X} \otimes \mathcal{Y}$ equipped with $\|\cdot\|_\varepsilon$.)*

Using the definition of the projective Banach space tensor product, it is not hard to see that $\mathcal{X}^*\widehat{\otimes}_\varepsilon\mathcal{Y}^*$ is isometrically isomorphic to a subspace of $(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^*$. We employ an argument similar to [56, p. 46]. If $u = \sum_{i=1}^n \varphi_i \otimes \psi_i \in \mathcal{X}^* \otimes \mathcal{Y}^*$, then there is a bilinear form $B_u \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$ given by

$$B_u(x, y) = \sum_{i=1}^n \varphi_i(x)\psi_i(y).$$

By definition of the injective tensor norm, we see that

$$\|u\|_\varepsilon = \sup\left\{|\alpha \otimes \beta(u)| : \alpha \in \mathcal{X}^{**},\ \beta \in \mathcal{Y}^{**},\ \|\alpha\| \le 1,\ \|\beta\| \le 1\right\}.$$

Using Goldstine's theorem [14, Theorem V.4.1], we may replace $\mathcal{X}^{**}$ by $\mathcal{X}$ and $\mathcal{Y}^{**}$ by $\mathcal{Y}$ in the supremum and obtain

$$\|u\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^n \varphi_i(x)\psi_i(y)\right| : x \in \mathcal{X},\ y \in \mathcal{Y},\ \|x\| \le 1,\ \|y\| \le 1\right\} = \|B_u\|.$$

Since $\|B_u\|$ is equal to the norm of the linearization $\widetilde{B}_u = u$ on $\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y}$, it follows that $\|u\|_\varepsilon = \|u\|_{(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^*}$ for all $u \in \mathcal{X} \otimes \mathcal{Y}$. Extending to the closure, we see that $\mathcal{X}^*\widehat{\otimes}_\varepsilon\mathcal{Y}^*$ is isometric to a subspace of $(\mathcal{X}\widehat{\otimes}_\pi\mathcal{Y})^*$.

The following theorem characterizes all reasonable cross-norms on $\mathcal{X} \otimes \mathcal{Y}$.

Theorem 1.2.5. (See [56, Proposition 6.1]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let α be a norm on $\mathcal{X} \otimes \mathcal{Y}$. Then α is a reasonable cross-norm if and only if $\|z\|_\varepsilon \le \alpha(z) \le \|z\|_\pi$ for all $z \in \mathcal{X} \otimes \mathcal{Y}$.*

If α is a reasonable cross-norm on $\mathcal{X} \otimes \mathcal{Y}$, then we let $\mathcal{X} \otimes_\alpha \mathcal{Y}$ denote the vector space $\mathcal{X} \otimes \mathcal{Y}$ equipped with the norm α, and we let $\mathcal{X}\widehat{\otimes}_\alpha\mathcal{Y}$ denote the completion of $\mathcal{X} \otimes_\alpha \mathcal{Y}$ with respect to α.

Suppose that α is a reasonable cross-norm defined for all pairs of Banach spaces. We say that α is **functorial** if, whenever $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ are Banach spaces and $S : \mathcal{X}_1 \to \mathcal{X}_2$ and $T : \mathcal{Y}_1 \to \mathcal{Y}_2$ are bounded linear maps, then the tensor product map $S \otimes T : \mathcal{X}_1 \otimes \mathcal{Y}_1 \to \mathcal{X}_2 \otimes \mathcal{Y}_2$ extends to a bounded linear map from $\mathcal{X}_1\widehat{\otimes}_\alpha\mathcal{Y}_1$ to $\mathcal{X}_2\widehat{\otimes}_\alpha\mathcal{Y}_2$ with norm at most $\|S\|\|T\|$. A helpful fact is the following.

Theorem 1.2.6. (See [56, p. 129]) *The injective and projective Banach space tensor norms are functorial.*

We will also need some background on integral operators and nuclear operators. The main result that we will need is that the integral norm and the nuclear norm coincide in finite dimensions. We proceed as in [56]. The motivation for nuclear operators is the following: if $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, then one can associate to each element of $\mathcal{X}^*\widehat{\otimes}_\pi\mathcal{Y}$ a bounded linear operator from $\mathcal{X}$ to $\mathcal{Y}$. Indeed, if $u = \sum_{n=1}^\infty \varphi_n \otimes y_n$ is an element of $\mathcal{X}^*\widehat{\otimes}_\pi\mathcal{Y}$, then we may define a linear operator $L_u : \mathcal{X} \to \mathcal{Y}$ such that

$$L_u(x) = \sum_{n=1}^\infty \varphi_n(x) y_n.$$

It is not hard to see that the map $J : \mathcal{X}^*\widehat{\otimes}_\pi\mathcal{Y} \to \mathcal{B}(\mathcal{X}, \mathcal{Y})$ defined by $J(u) = L_u$ is linear with $\|J\| = 1$. In general, J may not be surjective, but the operators in the range of J are referred to as the nuclear operators. In other words, if $T : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator, then we say that T is a **nuclear operator** if there is a sequence of functionals $(\varphi_n)_{n=1}^\infty \subset \mathcal{X}^*$ and a sequence of elements $(y_n)_{n=1}^\infty \subset \mathcal{Y}$ such that $\sum_{n=1}^\infty \|\varphi_n\|\|y_n\| < \infty$ and $T(x) = \sum_{n=1}^\infty \varphi_n(x)y_n$ for all $x \in \mathcal{X}$. If $T : \mathcal{X} \to \mathcal{Y}$ is a bounded linear operator that is nuclear, then the **nuclear norm** of T is given by

$$\|T\|_N = \inf\left\{\sum_{n=1}^\infty \|\varphi_n\|\|y_n\|\right\},$$

where the infimum is taken over all sequences $(\varphi_n)_{n=1}^\infty \subset \mathcal{X}^*$ and $(y_n)_{n=1}^\infty \subset \mathcal{Y}$ satisfying $\sum_{n=1}^\infty \|\varphi_n\|\|y_n\| < \infty$ and $T(x) = \sum_{n=1}^\infty \varphi_n(x)y_n$ for all $x \in \mathcal{X}$. We will let $\mathcal{N}(\mathcal{X}, \mathcal{Y})$ be the space of nuclear operators from $\mathcal{X}$ to $\mathcal{Y}$. It is evident that, if T is a nuclear operator, then $\|T\| \leq \|T\|_N$. If $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional, then $\mathcal{X}^*\widehat{\otimes}_\pi\mathcal{Y} = \mathcal{X}^* \otimes_\pi \mathcal{Y}$ is finite-dimensional, so that every nuclear operator T can be written in the form $T = \sum_{i=1}^n \varphi_n(x)y_n$ for functionals $\varphi_n \in \mathcal{X}^*$, elements $y_n \in \mathcal{Y}$, and a number $n \in \mathbb{N}$. It is not hard to see that, in this case, $\|T\|_N$ is the same as taking the infimum over all representations of T using *finite sums*.

Related to the class of nuclear operators is the class of integral operators. Integral operators relate to bounded linear functionals on the injective tensor product. To this end, the following proposition is helpful for our purposes.

Proposition 1.2.7. (See [56, Proposition 3.14]) *Let B be a bounded bilinear form on $\mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Let $\overline{b_1(\mathcal{X}^*)}$ and $\overline{b_1(\mathcal{Y}^*)}$ denote the closed unit balls in $\mathcal{X}^*$ and $\mathcal{Y}^*$, respectively. The linearization $\widetilde{B}$ of B belongs to $(\mathcal{X}\widehat{\otimes}_\varepsilon\mathcal{Y})^*$ if and only if there is a regular Borel measure μ on $\overline{b_1(\mathcal{X}^*)} \times \overline{b_1(\mathcal{Y}^*)}$ such that, for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,*

$$B(x, y) = \int_{\overline{b_1(\mathcal{X}^*)}\times\overline{b_1(\mathcal{Y}^*)}} \varphi(x)\psi(y)\, d\mu((\varphi, \psi)). \tag{1.2.1}$$

If this is the case, then the norm of $\widetilde{B}$ as a functional on $\mathcal{X}\widehat{\otimes}_\varepsilon\mathcal{Y}$ is given by $\|\widetilde{B}\| = \inf \|\mu\|$, where the infimum is taken over all regular Borel measures μ satisfying equation (1.2.1)*.*

If the linearization of a bilinear form $B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$ belongs to $(\mathcal{X}\widehat{\otimes}_\varepsilon\mathcal{Y})^*$, then we will call B an **integral bilinear form**, and its **integral norm** $\|B\|_I$ is the norm of the linearization of B as a functional on the injective tensor product.

For each operator $T : \mathcal{X} \to \mathcal{Y}$, we may consider the bilinear form $B_T : \mathcal{X} \times \mathcal{Y}^* \to \mathbb{C}$ given by $B_T(x, \varphi) = \varphi(Tx)$. We will say that T is an **integral operator** if B_T is an integral bilinear form, and we will define the **integral norm** of T to be $\|T\|_I = \|B_T\|_I$. We let $\mathcal{I}(\mathcal{X}, \mathcal{Y})$ denote the space of integral operators from $\mathcal{X}$ to $\mathcal{Y}$.

If $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional, then so is $\mathcal{X}^* \otimes_\pi \mathcal{Y}$. In particular, $\mathcal{X}^* \otimes_\pi \mathcal{Y}$ is reflexive. Since $\mathcal{X}$ and $\mathcal{Y}$ are finite-dimensional, the vector space dual $(\mathcal{X}^* \otimes \mathcal{Y})^*$ is canonically isomorphic to $\mathcal{X} \otimes \mathcal{Y}^*$ as a vector space. Thus, $(\mathcal{X}^* \otimes_\pi \mathcal{Y})^* \simeq \mathcal{X} \otimes_\varepsilon \mathcal{Y}^*$ isometrically. It follows that the dual of $\mathcal{X} \widehat{\otimes}_\varepsilon \mathcal{Y}^*$ is isometrically isomorphic to $\mathcal{X}^* \widehat{\otimes}_\pi \mathcal{Y}$. On the other hand, in the finite-dimensional setting, $\mathcal{N}(\mathcal{X}, \mathcal{Y}) = \mathcal{X}^* \widehat{\otimes}_\pi \mathcal{Y}$ isometrically [56, Corollary 4.8]. Combining these facts, we yield the desired result.

Theorem 1.2.8. (See [56, Corollary 4.17]) *Let $\mathcal{X}$ and $\mathcal{Y}$ be finite-dimensional normed spaces. Then $\mathcal{B}(\mathcal{X}, \mathcal{Y}) = \mathcal{N}(\mathcal{X}, \mathcal{Y}) = \mathcal{I}(\mathcal{X}, \mathcal{Y})$. Moreover, $\|T\|_N = \|T\|_I$ for all $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$.*

1.2.2 Tensor products of Hilbert spaces and the Schmidt decomposition

When considering Banach space tensor products of Hilbert spaces, there is one reasonable cross-norm that yields the structure of a Hilbert space. Indeed, if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then the inner product on the algebraic tensor product $\mathcal{H} \odot \mathcal{K}$ is given on simple tensors by

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle.$$

The inner product extends to all of $\mathcal{H} \odot \mathcal{K}$ by conjugate linearity. Then the completion of $\mathcal{H} \odot \mathcal{K}$ with respect to the norm induced by the inner product is a Hilbert space. For simplicity, we often let $\mathcal{H} \otimes \mathcal{K}$ denote the Hilbert space tensor product of $\mathcal{H}$ and $\mathcal{K}$. Given the definition of the inner product, it is easy to see that the Hilbert space norm on $\mathcal{H} \odot \mathcal{K}$ is a cross-norm. Moreover, an application of the Riesz representation theorem for Hilbert spaces shows that the norm on $\mathcal{H} \odot \mathcal{K}$ is a reasonable cross-norm. Thus, if $\|\cdot\|$ denotes the Hilbert space tensor product norm on $\mathcal{H} \otimes \mathcal{K}$, then $\|u\|_\varepsilon \leq \|u\| \leq \|u\|_\pi$ for all $u \in \mathcal{H} \odot \mathcal{K}$.

One tool that we will frequently use is the Schmidt decomposition in Hilbert space tensor products.

Theorem 1.2.9. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let $\xi \in \mathcal{H} \otimes \mathcal{K}$ be a unit vector. Then there exist orthonormal sequences $(u_i)_{i=1}^\infty \subseteq \mathcal{H}$ and $(v_i)_{i=1}^\infty \subseteq \mathcal{K}$ along with a unique non-increasing sequence of non-negative numbers $\alpha_1 \geq \alpha_2 \geq \dots$ such that $\sum_{i=1}^\infty |\alpha_i|^2 = 1$ and*

$$\xi = \sum_{i=1}^\infty \alpha_i u_i \otimes v_i.$$

(The sequence $(\alpha_i)_{i=1}^{\infty}$ is sometimes referred to as the sequence of ***Schmidt coefficients*** *for ξ.)*

Moreover, if $\zeta \in \mathcal{H} \otimes \mathcal{K}$ is another unit vector whose sequence of Schmidt coefficients is $(\alpha_i)_{i=1}^{\infty}$, then there are unitaries $U \in \mathcal{U}(\mathcal{H})$ and $V \in \mathcal{U}(\mathcal{K})$ such that $\xi = (U \otimes V)\zeta$.

In the case when $\mathcal{H}$ and $\mathcal{K}$ are finite-dimensional, only finitely many of the α_i's are non-zero. For $\xi \in \mathcal{H} \otimes \mathcal{K}$ with Schmidt decomposition sequence $(\alpha_i)_{i=1}^{\infty}$, the **Schmidt rank** of ξ is the number of elements of the Schmidt decomposition sequence that are non-zero.

Whenever $\xi \in \mathcal{H} \otimes \mathcal{K}$ and $U \in \mathcal{U}(\mathcal{H})$ and $V \in \mathcal{U}(\mathcal{K})$ are unitaries, then $(U \otimes V)\xi$ must have the same Schmidt coefficients as ξ. In particular, in the finite-dimensional case, the Schmidt rank of ξ is the same as the Schmidt rank of $(U \otimes V)\xi$. That is to say, Schmidt coefficients and Schmidt rank are invariant under "local unitaries" (i.e., tensor products of unitaries). We will use these facts freely in Chapters 3, 5 and 6.

1.2.3 Tensor products of C^*-algebras

One of the most important reformulations of Connes' embedding problem is Kirchberg's conjecture (see Conjecture 1.8.4), which asks whether there is a unique C^*-tensor product norm on the tensor product of $C^*(\mathbb{F}_\infty)$ with itself, where $C^*(\mathbb{F}_\infty)$ is the universal C^*-algebra of the free group on a countably infinite set of generators. As with Banach space tensor products, there is a largest C^*-tensor norm and a smallest C^*-tensor norm; we will only need these two tensor norms for our purposes.

The largest C^*-tensor norm is often referred to as the maximal tensor norm, which we define below. The maximal tensor norm can be thought of as the universal C^*-norm with respect to representations that have commuting ranges. If $\mathcal{X}$ and $\mathcal{Y}$ are vector spaces and $\varphi : \mathcal{X} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{Y} \to \mathcal{B}(\mathcal{H})$ are linear maps, then we say that φ and ψ have **commuting ranges** if $\varphi(x)\psi(y) = \psi(y)\varphi(x)$ for all $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. In this case, we define the product map $\varphi \cdot \psi : \mathcal{X} \otimes \mathcal{Y} \to \mathcal{B}(\mathcal{H})$ to be the linear map given on simple tensors by $(\varphi \cdot \psi)(x \otimes y) = \varphi(x)\psi(y)$.

Definition 1.2.10. *For C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the* ***maximal tensor product*** *of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \otimes_{\max} \mathcal{B}$, is the completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the norm given as follows: for $x \in \mathcal{A} \otimes \mathcal{B}$, we set*

$$\|x\|_{\max} = \sup\{\|\pi \cdot \rho(x)\|_{\mathcal{B}(\mathcal{H})}\},$$

where the supremum is taken over all $*$-homomorphisms $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $\rho : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ with commuting ranges.

It is well-known (see [8, Chapter 3]) that the max norm is the same as taking the supremum over all non-degenerate representations with commuting ranges. On the other hand, if $\mathcal{A}$ or $\mathcal{B}$ is unital, then one can arrange for the corresponding homomorphisms to be unital in the supremum.

There are many ways to define the minimal tensor product of C^*-algebras. For simplicity, we use the following definition.

Definition 1.2.11. *Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{B} \subseteq \mathcal{B}(\mathcal{K})$ be C^*-algebras. The* ***minimal tensor product*** *of $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A} \otimes_{\min} \mathcal{B}$, is the completion of the image of $\mathcal{A} \otimes \mathcal{B}$ in $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.*

The minimal tensor norm is also referred to as the **spatial norm**. It is well-known (see [8, Chapter 3]) that the minimal tensor product does not depend on the choice of Hilbert spaces $\mathcal{H}, \mathcal{K}$ or the choice of embeddings of $\mathcal{A}$ and $\mathcal{B}$. The minimal tensor product is also **injective**; that is, if $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1, \mathcal{B}_2$ are C^*-algebras and $\mathcal{A}_1 \subseteq \mathcal{A}_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$, then $\mathcal{A}_1 \otimes_{\min} \mathcal{B}_1$ is $*$-isomorphic to its inclusion in $\mathcal{A}_2 \otimes_{\min} \mathcal{B}_2$. In fact, the property of injectivity also holds for the minimal operator space tensor product. Since we only use this tensor product once in Chapter 5, we define it here. (For an excellent introduction to operator space tensor products, the reader is encouraged to see [6].)

Let X and Y be operator spaces. If $A = (a_{ij}) \in M_n(X)$ and $B = (b_{k\ell}) \in M_m(Y)$, then by $A \otimes B$ we mean the matrix $(a_{ij} \otimes b_{k\ell})_{(i,j),(k,\ell)} \in M_{nm}(X \otimes Y)$. An **operator space structure** on $X \otimes Y$ is a sequence of matrix norms $\| \cdot \|_n$ on $M_n(X \otimes Y)$ such that

- $X \otimes Y$ with the matrix norms $\| \cdot \|_n$ is a matricially normed space;
- $\|A \otimes B\|_{nm} \leq \|A\|_n \|B\|_m$ whenever $A \in M_n(X)$ and $B \in M_m(X)$; and
- Whenever $\varphi : X \to M_n$ and $\psi : Y \to M_m$ are completely bounded maps, then $\varphi \otimes \psi : X \otimes Y \to M_{nm}$ is completely bounded with $\|\varphi \otimes \psi\|_{cb} \leq \|\varphi\|_{cb} \|\psi\|_{cb}$.

There is a largest operator space tensor product, called the projective operator space tensor product (see [6]). The smallest operator space structure on $X \otimes Y$ is called the **injective tensor product**. The norm structure is defined as follows: if $X \subseteq \mathcal{B}(\mathcal{H})$ and $Y \subseteq \mathcal{B}(\mathcal{K})$ are completely isometric representations of X and Y, then the injective tensor product $X \check{\otimes} Y$ is the completion of $X \otimes Y$ with respect to the norm structure inherited from the inclusion $X \otimes Y \subseteq \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$. It follows by [6] that the injective operator space tensor product does not depend on the choice of embeddings of X and Y. In particular, whenever $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, we have $\mathcal{A} \check{\otimes} \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ completely isometrically.

The injective operator space tensor product is also injective, in the sense that, whenever $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ are operator spaces, then $X_1 \check{\otimes} Y_1$ is completely isometric to its inclusion in $X_2 \check{\otimes} Y_2$.

One fact that we will need in Chapter 5 relates the injective Banach space tensor product and the injective operator space tensor product for subspaces of a unital, commutative C^*-algebra. We recall that a unital commutative C^*-algebra $\mathcal{A}$ is isomorphic to $C(X)$, for some compact Hausdorff space X. It is well-known that, for any Banach space $\mathcal{Y}$, $C(X) \widehat{\otimes}_\varepsilon \mathcal{Y}$ is isometric to the Banach space $C(X, \mathcal{Y})$ of continuous $\mathcal{Y}$-valued functions on X [56, p. 49]. On the other hand, if $\mathcal{B}$ is a C^*-algebra, then $C(X) \otimes_{\min} \mathcal{B}$ is isomorphic to $C(X, \mathcal{B})$ [49, Proposition 12.5]. By injectivity of the tensor products in question, we obtain the following theorem:

Theorem 1.2.12. *Let $\mathcal{A}$ be a unital commutative C^*-algebra, and let $\mathcal{B}$ be a C^*-algebra. Then $\mathcal{A} \widehat{\otimes}_\varepsilon \mathcal{B} = \mathcal{A} \otimes_{\min} \mathcal{B}$ isometrically. Moreover, if $E \subseteq \mathcal{A}$ is an operator space and $F \subseteq \mathcal{B}$ is an operator space, then*

$$E \widehat{\otimes}_\varepsilon F = E \check{\otimes} F$$

isometrically.

1.2.4 Tensor products of operator systems

In this section we define the various operator system tensor products that we shall use. We follow the terminology of [38]. If $\mathcal{S}$ and $\mathcal{T}$ are operator systems, then the vector space $\mathcal{S} \otimes \mathcal{T}$ becomes a complex $*$-vector space with the involution $*$ given on simple tensors by $(s \otimes t)^* = s^* \otimes t^*$. An **operator system structure** on the vector space tensor product $\mathcal{S} \otimes \mathcal{T}$ is a sequence of cones $\{C_n\}_{n=1}^\infty$, where $C_n \subseteq (M_n(\mathcal{S} \otimes \mathcal{T}))_h$, such that

- $(\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ is an operator system;
- If $n, m \in \mathbb{N}$, $X \in M_n(\mathcal{S})_+$ and $Y \in M_m(\mathcal{T})_+$, then $X \otimes Y := (x_{ij} \otimes y_{k\ell})_{(i,j),(k,\ell)}$ is in C_{nm}, and
- If $\varphi : \mathcal{S} \to M_n$ is ucp and $\psi : \mathcal{T} \to M_m$ is ucp, then the tensor product map $\varphi \otimes \psi : (\mathcal{S} \otimes \mathcal{T}, \{C_n\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}}) \to M_{nm}$ is ucp.

An **operator system tensor product** is a map τ sending any pair of operator systems $(\mathcal{S}, \mathcal{T})$ to an operator system structure $\tau(\mathcal{S}, \mathcal{T})$ on $\mathcal{S} \otimes \mathcal{T}$, which we will denote by $\mathcal{S} \otimes_\tau \mathcal{T}$. Unlike tensor products of C^*-algebras, we will generally deal with the operator system structure on the vector space $\mathcal{S} \otimes \mathcal{T}$, rather than its completion.

An operator system tensor product τ is said to be **symmetric** if, for every pair of operator systems $(\mathcal{S}, \mathcal{T})$, the flip map $s \otimes t \mapsto t \otimes s$ is a complete order isomorphism between $\mathcal{S} \otimes_\tau \mathcal{T}$ and $\mathcal{T} \otimes_\tau \mathcal{S}$. An operator system tensor product τ is **functorial** if, whenever $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ are operator systems and $\varphi_1 : \mathcal{S}_1 \to \mathcal{T}_1$ and $\varphi_2 : \mathcal{S}_2 \to \mathcal{T}_2$ are ucp, then $\varphi_1 \otimes \varphi_2 : \mathcal{S}_1 \otimes_\tau \mathcal{S}_2 \to \mathcal{T}_1 \otimes_\tau \mathcal{T}_2$ is ucp.

An important fact about operator system tensor products is that they are a special subclass of operator space tensor products.

Proposition 1.2.13. (Kavruk-Paulsen-Todorov-Tomforde, [38, Proposition 3.4]) *Let τ be an operator system tensor product. Then τ is an operator space tensor product. In other words:*

1. *If $\mathcal{S}$ and $\mathcal{T}$ are operator systems and $X \in M_n(\mathcal{S})$ and $Y \in M_m(\mathcal{T})$, then*

$$\|X \otimes Y\|_{M_{nm}(\mathcal{S} \otimes_\tau \mathcal{T})} \leq \|X\|_{M_n(\mathcal{S})} \|Y\|_{M_m(\mathcal{T})}.$$

2. *If $\varphi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are completely bounded maps, then $\varphi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{nm}$ is completely bounded with $\|\varphi \otimes \psi\|_{cb} \leq \|\varphi\|_{cb} \|\psi\|_{cb}$.*

If α, β are two operator system tensor products, then we write $\alpha \leq \beta$ if, for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, the identity map $\text{id} : \mathcal{S} \otimes_\beta \mathcal{T} \to \mathcal{S} \otimes_\alpha \mathcal{T}$ is ucp. We write $\alpha = \beta$ if $\alpha \leq \beta$ and $\beta \leq \alpha$. For a pair of operator systems $(\mathcal{S}, \mathcal{T})$ and a pair of operator system tensor products α and β, we will write $\mathcal{S} \otimes_\alpha \mathcal{T} = \mathcal{S} \otimes_\beta \mathcal{T}$ if the identity map is a complete order isomorphism between $\mathcal{S} \otimes_\alpha \mathcal{T}$ and $\mathcal{S} \otimes_\beta \mathcal{T}$. Given an operator system $\mathcal{S}$ and operator system tensor products α, β with $\alpha \leq \beta$, we say that $\mathcal{S}$ is (α, β)-**nuclear** if, for every operator system $\mathcal{T}$, we have $\mathcal{S} \otimes_\alpha \mathcal{T} = \mathcal{S} \otimes_\beta \mathcal{T}$.

Below, we define the various operator system tensor products that we shall need. The smallest operator system tensor product is called the minimal tensor product.

Definition 1.2.14. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. The* ***minimal tensor product*** *of $\mathcal{S}$ and $\mathcal{T}$, denoted $\mathcal{S} \otimes_{\min} \mathcal{T}$, is defined by the positive cones $\{\mathcal{C}_p^{\min}(\mathcal{S}, \mathcal{T})\}_{p=1}^\infty$ satisfying the rule that $X \in M_p(\mathcal{S} \otimes \mathcal{T})$ belongs to $\mathcal{C}_p^{\min}(\mathcal{S}, \mathcal{T})$ if and only if, whenever $\varphi : \mathcal{S} \to M_n$ and $\psi : \mathcal{T} \to M_m$ are ucp maps, then $(\varphi \otimes \psi)^{(p)}(X) \in M_p(M_{mn})_+$.*

A useful fact about the minimal tensor product of operator systems is that it coincides with the injective operator space tensor product. In other words, for any operator systems $\mathcal{S}$ and $\mathcal{T}$, the identity map is a complete isometry between $\mathcal{S} \otimes_{\min} \mathcal{T}$ and $\mathcal{S} \check{\otimes} \mathcal{T}$ [38, Corollary 4.9]. In particular, because of this fact, it readily follows that the minimal operator system tensor product is injective. That is to say, whenever $\mathcal{S}_1 \subseteq \mathcal{S}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ are operator systems, then $\mathcal{S}_1 \otimes_{\min} \mathcal{T}_1$ is completely order isomorphic to the operator system obtained from the inclusion of $\mathcal{S}_1 \otimes \mathcal{T}_1$ into $\mathcal{S}_2 \otimes_{\min} \mathcal{T}_2$. Moreover, if $\mathcal{A}$ and $\mathcal{B}$ are unital C^*-algebras, then the operator system tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$ is completely order isomorphic to the image of $\mathcal{A} \otimes \mathcal{B}$ in $\mathcal{A} \otimes_{C^*-\min} \mathcal{B}$ [38, Corollary 4.10]. For this reason, we will always use $\otimes_{\min}$ for the minimal tensor product of (unital) C^*-algebras or the minimal tensor product of operator systems.

The largest operator system tensor product is referred to as the maximal tensor product.

Definition 1.2.15. *For operator system tensor products $\mathcal{S}$ and $\mathcal{T}$, we define*

$$\mathcal{D}_p^{\max}(\mathcal{S}, \mathcal{T}) = \{A(X \otimes Y)A^* : X \in M_m(\mathcal{S})_+,\, Y \in M_n(\mathcal{S})_+,\, A \in M_{p,mn}\}.$$

The ***maximal tensor product*** *of $\mathcal{S}$ and $\mathcal{T}$, denoted $\mathcal{S} \otimes_{\max} \mathcal{T}$, is defined by the positive cones*

$$\mathcal{C}_p^{\max}(\mathcal{S}, \mathcal{T}) := \{X \in M_p(\mathcal{S} \otimes \mathcal{T})_h : X + \varepsilon I_p \in \mathcal{D}_p^{\max} \text{ for all } \varepsilon > 0\}.$$

In finite dimensions, the minimal and maximal tensor products are dual to each other. That is to say, if $\mathcal{S}$ and $\mathcal{T}$ are finite-dimensional operator systems, then $(\mathcal{S} \otimes_{\min} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{\max} \mathcal{T}^d$, and $(\mathcal{S} \otimes_{\max} \mathcal{T})^d$ is completely order isomorphic to $\mathcal{S}^d \otimes_{\min} \mathcal{T}^d$ [24].

Analogous to the maximal C^*-tensor product, one can consider an operator system tensor product that is universal with respect to commuting pairs of ucp maps.

Definition 1.2.16. *For operator systems $\mathcal{S}$ and $\mathcal{T}$, the* ***commuting tensor product*** *of $\mathcal{S}$ and $\mathcal{T}$, denoted $\mathcal{S} \otimes_c \mathcal{T}$, is defined by the positive cones $\mathcal{C}_p^{comm}(\mathcal{S}, \mathcal{T})$, which satisfy the rule that $X \in M_p(\mathcal{S} \otimes \mathcal{T})_h$ belongs to $\mathcal{C}_p^{comm}(\mathcal{S}, \mathcal{T})$ if and only if, whenever $\varphi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{T} \to \mathcal{B}(\mathcal{H})$ are ucp maps with commuting ranges, then $(\varphi \cdot \psi)^{(p)}(X) \in M_p(\mathcal{B}(\mathcal{H}))_+$.*

Each of the minimal, maximal and commuting tensor products are symmetric, functorial tensor products [38]. Similar to the case for the minimal tensor product, the commuting tensor product of unital C^*-algebras coincides with the maximal C^*-tensor product. In fact, the following is true.

Theorem 1.2.17. (Kavruk-Paulsen-Todorov-Tomforde, [38, Theorem 6.7]) *Every unital C^*-algebra is $(c, \max)$-nuclear.*

The last two tensor products that we need are examples of asymmetric tensor products.

Definition 1.2.18. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. We define the* ***essential left*** *tensor product, denoted $\mathcal{S} \otimes_{el} \mathcal{T}$, to be the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ when considered as a subspace of $\mathcal{I}(\mathcal{S}) \otimes_{\max} \mathcal{T}$.*

Definition 1.2.19. *For operator systems $\mathcal{S}$ and $\mathcal{T}$, the* ***essential right*** *tensor product $\mathcal{S} \otimes_{er} \mathcal{T}$ is the operator system obtained from the inclusion $\mathcal{S} \otimes \mathcal{T} \subseteq \mathcal{S} \otimes_{\max} \mathcal{I}(\mathcal{T})$.*

It is not hard to see that the flip map $s \otimes t \mapsto t \otimes s$ extends to a complete order isomorphism $\mathcal{S} \otimes_{\text{el}} \mathcal{T} \simeq \mathcal{T} \otimes_{\text{er}} \mathcal{S}$; similarly, $\mathcal{S} \otimes_{\text{er}} \mathcal{T} \simeq \mathcal{T} \otimes_{\text{el}} \mathcal{S}$.

We close this section by summarizing the known relations between each of the operator system tensor products defined above. It was shown in [38] that the following chain of inequalities holds:

$$\min \leq \text{el}, \text{er} \leq c \leq \max .$$

1.3 Almost finite-dimensional properties of operator systems and C^*-algebras

Many of the different formulations of Connes' embedding problem involve approximations from a finite-dimensional setting. From the perspective of C^*-algebras and operator systems, certain properties of this form arise. In this section, we will introduce the approximation properties of C^*-algebras and operator systems that pertain to the emebdding problem. In particular, we will consider the local lifting property, the weak expectation property, and certain operator system variants of these properties. We will end this section by summarizing the background results required on residually finite-dimensional C^*-algebras and quasidiagonal C^*-algebras. For more information on the interaction between operator systems and properties like the local lifting property or the weak expectation property, see [37]. The reader is referred to [8, Chapter 7] for an introduction to quasidiagonal and residually finite-dimensional C^*-algebras.

Definition 1.3.1. *An operator system $\mathcal{S}$ has the* ***operator system local lifting property*** *(abbreviated* ***OSLLP****) if, whenever $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{I}$ is an ideal in $\mathcal{A}$ and $\varphi : \mathcal{S} \to \mathcal{A}/\mathcal{I}$ is a ucp map, then for each finite-dimensional subsystem $\mathcal{F} \subseteq \mathcal{S}$, there is a ucp map $\psi_{\mathcal{F}} : \mathcal{F} \to \mathcal{A}$ such that $q \circ \psi_{\mathcal{F}} = \varphi_{|\mathcal{F}}$, where $q : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ is the canonical quotient map.*

If a unital C^*-algebra $\mathcal{C}$ has the OSLLP, then we will simply write that $\mathcal{C}$ has the **local lifting property** (abbreviated **LLP**).

Definition 1.3.2. *An operator system $\mathcal{S}$ has the* ***weak expectation property*** *(abbreviated* ***WEP****) if there is a ucp map $\varphi : \mathcal{I}(\mathcal{S}) \to \mathcal{S}^{dd}$ such that $\varphi(s) = \iota(s)$ for all $s \in \mathcal{S}$, where $\iota : \mathcal{S} \to \mathcal{S}^{dd}$ is the canonical inclusion map.*

A property related to the WEP (but weaker than the WEP) is the double commutant expectation property.

Definition 1.3.3. *An operator system $\mathcal{S}$ has the* ***double commutant expectation property*** *(abbreviated* ***DCEP****) if, whenever $\iota : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is a unital complete order embedding, there is a ucp map $\varphi : \mathcal{I}(\mathcal{S}) \to \iota(\mathcal{S})''$ extending ι.*

We can characterize these properties by certain forms of nuclearity.

Theorem 1.3.4. (Kavruk-Paulsen-Todorov-Tomforde, [39, Theorem 8.5]) *Let $\mathcal{S}$ be an operator system. The following statements are equivalent:*

1. *$\mathcal{S}$ has the OSLLP.*
2. *$\mathcal{S}$ is $(\min, er)$-nuclear.*
3. *$\mathcal{S} \otimes_{\min} \mathcal{B}(\mathcal{H}) = \mathcal{S} \otimes_{\max} \mathcal{B}(\mathcal{H})$ for every Hilbert space $\mathcal{H}$.*

The WEP relates to $(\mathrm{el}, \max)$-nuclearity.

Theorem 1.3.5. *Let $\mathcal{S}$ be an operator system. The following statements are equivalent:*

1. *$\mathcal{S}$ has the WEP.*
2. *$\mathcal{S}$ is $(el, \max)$-nuclear.*

The proof that (1) implies (2) is found in [39, Theorem 6.7], while the converse is shown in [28].

For the DCEP, an alternative characterization is in terms of the full group C^*-algebra $C^*(\mathbb{F}_\infty)$.

Theorem 1.3.6. (Kavruk-Paulsen-Todorov-Tomforde, [39, Theorem 7.6]) *For an operator system $\mathcal{S}$, the following are equivalent:*

1. *$\mathcal{S}$ has the DCEP.*

2. *$\mathcal{S}$ is (el, c)-nuclear.*

3. *$\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty)$.*

If $\mathcal{A}$ is a unital C^*-algebra, then by Theorem 1.2.17, $\mathcal{A}$ is (el, c) nuclear if and only if it is $(\mathrm{el}, \max)$ nuclear. Hence, the WEP and the DCEP are identical for unital C^*-algebras. In contrast, the WEP is in general a stronger property than the DCEP for operator systems. Indeed, if $n \geq 3$ and $\mathcal{T}_n$ is the operator system of tridiagonal matrices in M_n, then $\mathcal{T}_n$ is $(\min, c)$-nuclear but $\mathcal{T}_n \otimes_{\min} \mathcal{T}_n^d \neq \mathcal{T}_n \otimes_{\max} \mathcal{T}_n^d$ [38]. In particular, $\mathcal{T}_n$ is not $(c, \max)$-nuclear, so that $\mathcal{T}_n$ is not $(\mathrm{el}, \max)$-nuclear. Hence, $\mathcal{T}_n$ does not have the WEP. However, since $\mathcal{T}_n$ is $(\min, c)$-nuclear and $\min \leq \mathrm{el} \leq c$, it follows that $\mathcal{T}_n$ is (el, c)-nuclear. Hence, $\mathcal{T}_n$ has the DCEP.

The relevance of these nuclearity-related properties for operator systems and unital C^*-algebras is that they provide very useful versions of Connes' embedding problem.

Theorem 1.3.7. (Kavruk-Paulsen-Todorov-Tomforde, [39, Theorem 9.1]) *The following are equivalent.*

1. *$C^*(\mathbb{F}_\infty)$ has the WEP.*

2. *Every $(\min, er)$-nuclear operator system is (el, c)-nuclear.*

3. *Every $(\min, er)$-nuclear operator system $\mathcal{S}$ satisfies $\mathcal{S} \otimes_{\min} \mathcal{S} = \mathcal{S} \otimes_c \mathcal{S}$.*

4. *Every operator system with the OSLLP has the DCEP.*

There are several examples of operator systems that "detect" whether or not $C^*(\mathbb{F}_\infty)$ has the WEP, in terms of condition (3) of Theorem 1.3.7. A relevant example is the operator system $\mathcal{S}_n$ generated by the n universal unitary generators of $C^*(\mathbb{F}_n)$. Indeed, it is known that $C^*(\mathbb{F}_\infty)$ has the WEP if and only if $\mathcal{S}_n \otimes_{\min} \mathcal{S}_n = \mathcal{S}_n \otimes_c \mathcal{S}_n$ for all (equivalently, some) $n \geq 2$ [37, Theorem 5.10]. Another example is the non-commuting cube $NC(n)$, which is the universal operator system generated by n self-adjoint contractions. Equivalently, it is the operator system spanned by the generators of each copy of $\mathbb{Z}_2$ in $C^*(*_n\mathbb{Z}_2)$ (see [22]).

A natural question for these kinds of operator systems is whether or not $\min = \max$ for tensor products of the operator system with itself. If this were true for one of the operator systems $\mathcal{S}_n$, then $C^*(\mathbb{F}_\infty)$ would have the WEP, which is equivalent to Kirchberg's conjecture that $C^*(\mathbb{F}_\infty) \otimes C^*(\mathbb{F}_\infty)$ has a unique C^*-norm [41]. However, all the examples of DCEP detecting operator systems $\mathcal{S}$ known satisfy $\mathcal{S} \otimes_c \mathcal{S} \neq \mathcal{S} \otimes_{\max} \mathcal{S}$, implying that $\mathcal{S} \otimes_{\min} \mathcal{S} \neq \mathcal{S} \otimes_{\max} \mathcal{S}$. The example most pertinent to our work is the operator system $\mathcal{S}_n$ from $C^*(\mathbb{F}_n)$.

Theorem 1.3.8. (Farenick-Kavruk-Paulsen-Todorov, [22, Theorem 3.8]) *For each $n, m \geq 2$, we have $\mathcal{S}_n \otimes_c \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$.*

One feature of many C^*-algebras that arise in statements of Kirchberg's conjecture is that they have certain finite-dimensional approximation properties. Two of these properties are residual finite-dimensionality and quasidiagonality, which we will outline below.

Definition 1.3.9. *A C^*-algebra $\mathcal{A}$ is said to be **residually finite-dimensional** (abbreviated **RFD**) if there is a collection $\{\pi_i\}_{i\in I}$ of finite-dimensional representations $\pi_i : \mathcal{A} \to M_{n(i)}$ such that $\bigoplus_{i\in I} \pi_i$ is faithful.*

A well-known fact is that if $\mathcal{A}_1$ and $\mathcal{A}_2$ are RFD, then so is their minimal tensor product $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ (see [8]).

A weaker notion than residual finite-dimensionality is quasidiagonality.

Definition 1.3.10. *A C^*-algebra $\mathcal{A}$ is said to be **quasidiagonal** (abbreviated **QD**) if there is a net $(\varphi_\lambda)_{\lambda\in\Lambda}$ of ccp maps $\varphi_\lambda : \mathcal{A} \to M_{k(\lambda)}$ such that, for all $a, b \in \mathcal{A}$,*

- $\lim_\lambda \|\varphi_\lambda(ab) - \varphi_\lambda(a)\varphi_\lambda(b)\| = 0$, *and*
- $\lim_\lambda \|\varphi_\lambda(a)\| = \|a\|$.

If $\mathcal{A}$ is a unital C^*-algebra that is QD, then the net of ccp maps in Definition 1.3.10 can be taken to be ucp maps [8, Lemma 7.1.4]. Many of the C^*-algebras arising in statements of Kirchberg's conjecture turn out to be RFD. In particular, $C^*(\mathbb{F}_n)$ is RFD [9] for every $n \geq 2$. We will see another example of such an algebra in Chapter 2.

There are many properties that relate to quasidiagonal algebras; see [8, Chapter 7] for a thorough introduction to this topic. The only property that we will need is that quasidiagonality is preserved by homotopy equivalence.

If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebra and $\pi, \rho : \mathcal{A} \to \mathcal{B}$ are $*$-homomorphisms, then π and ρ are **homotopic** if there is a family $(\pi_t)_{t\in[0,1]}$ of $*$-homomorphisms $\pi_t : \mathcal{A} \to \mathcal{B}$ such that $\pi_0 = \pi$, $\pi_1 = \rho$, and for each $a \in \mathcal{A}$, the map $t \mapsto \pi_t(a)$ is continuous. One useful relation between homotopic $*$-homomorphisms and quasidiagonality is a theorem of D. Voiculescu, which says that quasidiagonality is a homotopy invariant. For our purposes, we only need the following weaker statement.

Theorem 1.3.11. (Voiculescu, [67], see [8, Proposition 7.3.5]) *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras, and let $\pi_0, \pi_1 : \mathcal{A} \to \mathcal{B}$ be homotopic $*$-homomorphisms such that π_0 is injective and $\pi_1(\mathcal{A})$ is a QD subalgebra of $\mathcal{B}$. Then $\mathcal{A}$ is QD.*

1.4 Group embeddings

One very useful tool in the study of (matrix-valued) quantum correlations is that of group embeddings. In this section, we will exhibit some examples of how certain free product groups embed into other free product groups. In Chapters 5 and 6, we will use these embeddings to obtain separations in the different models for matrix-valued correlations. The reason that these group embeddings are so useful is that the universal C^*-algebra of a subgroup H of a discrete group G is contained faithfully in the universal C^*-algebra of G.

Proposition 1.4.1. (See [53, Proposition 8.8]) *Let G be a discrete group and let H be a subgroup of G. Then there is a $*$-homomorphism $\Phi : C^*(H) \to C^*(G)$ and a ucp map $\Psi : C^*(G) \to C^*(H)$ such that $\Psi \circ \Phi = id_{C^*(H)}$.*

To obtain our desired group embeddings, the main tool that we will use is often referred to as the "Ping-Pong lemma".

Lemma 1.4.2. (See [16, p. 25]) *Let G be a group acting on a set X, and let G_1, G_2 be subgroups of G. Let Γ be the subgroup of G generated by G_1 and G_2. Suppose that there are non-empty subsets $X_1, X_2 \subseteq X$ such that $X_1 \cap X_2 = \emptyset$ and*

$$g(X_2) \subseteq X_1, \;\; \forall g \in G_1 \setminus \{1\},$$
$$g(X_1) \subseteq X_2, \;\; \forall g \in G_2 \setminus \{1\}.$$

Then Γ is isomorphic to the free product group $G_1 * G_2$.

While all of the group embeddings that we will use are simple to demonstrate; we include the proofs for convenience. The next two propositions give embeddings of the free product group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ into the groups $*_4\mathbb{Z}_2$ and $*_3\mathbb{Z}_2$, respectively.

Proposition 1.4.3. *Let $\sigma_0, ..., \sigma_3$ be generators of $\mathbb{Z}_2$ and g be a generator of $\mathbb{Z}$. Then there is an injective group homomorphism $\iota : \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z} \to *_4\mathbb{Z}_2$ such that $\iota(\sigma_0) = \sigma_0$, $\iota(\sigma_1) = \sigma_1$ and $\iota(g) = \sigma_2\sigma_3$.*

Proof. Since σ_2 and σ_3 have no relations between each other, $\sigma_2\sigma_3$ has infinite order. By definition of $*_4\mathbb{Z}_2$, there are no relations between σ_0 and $\sigma_2\sigma_3$. Similarly, there are no relations between σ_1 and $\sigma_2\sigma_3$. It easily follows that the subgroup of $*_4\mathbb{Z}_2$ generated by σ_0, σ_1 and $\sigma_2\sigma_3$ is isomorphic to $\langle\sigma_0\rangle * \langle\sigma_1\rangle * \langle\sigma_2\sigma_3\rangle$, which is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$. $\square$

The embedding of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ into $*_3\mathbb{Z}_2$ requires the generator of $\mathbb{Z}$ to be sent to a longer word in the generators of $*_3\mathbb{Z}_2$.

Proposition 1.4.4. *Let $\sigma_0, \sigma_1, \sigma_2$ be a universal set of generators of $*_3\mathbb{Z}_2$, and let g be a generator of $\mathbb{Z}$. There is an injective group homomorphism $\iota : \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z} \to *_3\mathbb{Z}_2$ such that $\iota(\sigma_0) = \sigma_0$, $\iota(\sigma_1) = \sigma_1$ and $\iota(g) = \sigma_2\sigma_0\sigma_1\sigma_2$.*

Proof. Since $\sigma_0\sigma_1$ has infinite order in $*_3\mathbb{Z}_2$ and $\iota(g)$ is the conjugation of $\sigma_0\sigma_1$ by the order 2 element σ_2, we see that $\sigma_2\sigma_0\sigma_1\sigma_2$ has infinite order. Now, we let

$$X_1 = \{w \in *_3\mathbb{Z}_2 : w \text{ starts with } \sigma_0 \text{ or } \sigma_1\}$$
$$X_2 = \{w \in *_3\mathbb{Z}_2 : w \text{ starts with } \sigma_2\}.$$

Let $G_1 = \langle\sigma_0, \sigma_1\rangle = \mathbb{Z}_2 * \mathbb{Z}_2$, and let $G_2 = \langle\sigma_2\sigma_0\sigma_1\sigma_2\rangle$. Then for $g \in G_1 \setminus \{1\}$, the last letter in g must be either σ_0 or σ_1, and the first letter must be either σ_0 or σ_1. Hence, $g(X_2) \subseteq X_1$ whenever $g \in G_1 \setminus \{1\}$. Similarly, since there are no relations between σ_2 and $\langle\sigma_0, \sigma_1\rangle$, we see that $gX_2 \subseteq X_1$ whenever $g \in \langle\sigma_2\sigma_0\sigma_1\sigma_2\rangle \setminus \{1\}$. By Lemma 1.4.2, the subgroup of $*_3\mathbb{Z}_2$ generated by G_1 and G_2 is $G_1 * G_2$. Since $G_1 = \mathbb{Z}_2 * \mathbb{Z}_2$ and $G_2 \simeq \mathbb{Z}$, the map ι is indeed an injective group homomorphism from $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ to $*_3\mathbb{Z}_2$. $\square$

When the embeddings are into groups involving free products of $\mathbb{Z}_3$, the embeddings become more complicated. First, we give an embedding of $\mathbb{Z}_2 * \mathbb{Z}$ into $\mathbb{Z}_2 * \mathbb{Z}_3$, which we will use in Chapter 6.

Proposition 1.4.5. *Let g be the generator of $\mathbb{Z}_2$; let h be a generator of $\mathbb{Z}_3$; and let u be a generator of $\mathbb{Z}$. Then there is an injective group homomorphism $\iota : \mathbb{Z}_2 * \mathbb{Z} \hookrightarrow \mathbb{Z}_2 * \mathbb{Z}_3$ such that $\iota(g) = g$ and $\iota(u) = hgh$.*

Proof. Note that hgh has infinite order since h is order 3 and powers of hgh do not decrease in word length. On the other hand, let

$$X_1 = \{w \in \mathbb{Z}_2 * \mathbb{Z}_3 : w \text{ starts with either } h \text{ or } h^2\}$$

and

$$X_2 = \{w \in \mathbb{Z}_2 * \mathbb{Z}_3 : w \text{ starts with } g\},$$

where we consider words in reduced form. Clearly $gX_1 \subseteq X_2$ and $hX_2 \subseteq X_1$. Thus, $hghX_2 \subseteq hgX_1 \subseteq hX_2 \subseteq X_1$. By Lemma 1.4.2, the map $\iota : \mathbb{Z}_2 * \mathbb{Z} \to \mathbb{Z}_2 * \mathbb{Z}_3$ given by $\iota(g) = g$ and $\iota(u) = hgh$ extends to an injective group homomorphism. $\square$

The most complicated embedding that we shall need is the following embedding of $\mathbb{F}_3$ into $\mathbb{Z}_3 * \mathbb{Z}_3$.

Proposition 1.4.6. *Let g_0, g_1, g_2 be a set of universal generators of $\mathbb{F}_3$. Let a, b be generators of $\mathbb{Z}_3$. Then there is an injective group homomorphism $\iota : \mathbb{F}_3 \to \mathbb{Z}_3 * \mathbb{Z}_3$ given by $\iota(g_0) = aba$, $\iota(g_1) = bab$ and $\iota(g_2) = ab^2a^2b$.*

Proof. Since a has order 3, it is not hard to see that $\iota(g_0) = aba$ must have infinite order, since the word length does not decrease when taking powers of $\iota(g_0)$. Similarly, since b has order 3, the order of $\iota(g_1) = bab$ must be infinite. Since the first letter of $\iota(g_2) = ab^2a^2b$ is a and the last letter of $\iota(g_2)$ is b, it follows that $\iota(g_2)$ has infinite order as well. Let $G_1 = \langle aba \rangle$, $G_2 = \langle bab \rangle$ and $G_3 = \langle ab^2a^2b \rangle$. Now set

$$X_1 = \{w \in \mathbb{Z}_3 * \mathbb{Z}_3 : w \text{ starts with } aba\}$$
$$X_2 = \{w \in \mathbb{Z}_3 * \mathbb{Z}_3 : w \text{ starts with } bab\}$$

Since there are no relations between a and b, it is evident that $gX_1 \subseteq X_2$ whenever $g \in G_2 \setminus \{1\}$, while $gX_2 \subseteq X_1$ whenever $g \in G_1 \setminus \{1\}$. Since aba and bab have infinite order, Lemma 1.4.2 forces $\mathbb{F}_2 = \mathbb{Z} * \mathbb{Z} \simeq \langle aba, bab \rangle$. Next, we let

$$X_3 = \{w \in \mathbb{Z}_3 * \mathbb{Z}_3 : w \text{ starts with } ab^2\}.$$

Since $G_1 * G_2 \simeq \langle aba, bab \rangle \simeq \mathbb{F}_2$, whenever $g \in G_1 * G_2 \setminus \{1\}$, we have that $g(X_3) \subseteq X_1 \cup X_2$. If $g \in G_3 \setminus \{1\} = \langle ab^2a^2b \rangle \setminus \{1\}$, then since multiplying ab^2a^2b on the right by aba or bab does not decrease word length, we have that $g(X_1 \cup X_2) \subseteq X_2$. Applying Lemma 1.4.2 again, it follows that the subgroup of $\mathbb{Z}_3 * \mathbb{Z}_3$ generated by $G_1 * G_2$ and G_3 is isomorphic to $G_1 * G_2 * G_3$. Since $G_3 \simeq \mathbb{Z}$, it follows that this subgroup is isomorphic to $*_3\mathbb{Z} = \mathbb{F}_3$. Hence, the map ι is an injective group homomorphism, as desired. $\square$

1.5 Crossed products of C^*-algebras

In this section, we outline some of the concepts and results relating to crossed products of C^*-algebras arising from group actions. More information on crossed products can be found in [8, Chapter 4]. We will consider this construction in Chapter 5 in the context of certain universal C^*-algebras.

For simplicity, we will consider the case where $\mathcal{A}$ is a unital C^*-algebra and G is a countable discrete group. A **group action** of G on $\mathcal{A}$ is a homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, where $\mathrm{Aut}(\mathcal{A})$ is the group of unital $*$-isomorphisms of $\mathcal{A}$ onto itself. For $g \in G$, the element $\alpha(g) \in \mathrm{Aut}(\mathcal{A})$ is often written as α_g.

For a group action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$, we define $C_c(G, \mathcal{A})$ to be the space of all finitely supported functions from G into $\mathcal{A}$. If $a \in \mathcal{A}$ and $g \in G$, then we will denote by ag the function

$$ag(h) = \begin{cases} a & h = g \\ 0 & h \neq g. \end{cases}$$

In particular, $C_c(G, \mathcal{A}) = \mathrm{span}\,\{ag : a \in \mathcal{A},\, g \in G\}$. We equip $C_c(G, \mathcal{A})$ with a product as follows: if $x = \sum_{s \in G} a_s s$ and $y = \sum_{t \in G} b_t t$ are elements of $C_c(G, \mathcal{A})$, then we set

$$xy = \sum_{s,t \in G} a_s \alpha_s(b_t) st.$$

In other words, the product is defined such that, for $a \in \mathcal{A}$ and $s \in G$, we have

$$sas^{-1} = \alpha_s(a).$$

The involution on $C_c(G, \mathcal{A})$ is defined by

$$x^* = \sum_{s \in G} \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

This definition is easily seen to be the conjugate linear extension of the assignment

$$(a_s s)^* = \alpha_{s^{-1}}(a_s^*) s^{-1}.$$

With this product and involution, $C_c(G, \mathcal{A})$ is a $*$-algebra. Since $\mathcal{A}$ is unital and G is discrete, $C_c(G, \mathcal{A})$ has unit given by $1_{\mathcal{A}} e$, where e is the identity of G.

To consider the different C^*-norms on $C_c(G, \mathcal{A})$, it is helpful to consider the valid representations of $C_c(G, \mathcal{A})$. A **covariant representation** of $C_c(G, \mathcal{A})$ is a triple $(u, \pi, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, $u : G \to \mathcal{B}(\mathcal{H})$ is a unitary representation, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a $*$-homomorphism, such that for all $s \in G$ and $a \in \mathcal{A}$,

$$u_s \pi(a) u_s^* = \pi(\alpha_s(a)).$$

If $(u, \pi, \mathcal{H})$ is a covariant representation of $C_c(G, \mathcal{A})$, then we obtain a $*$-homomorphism $u \times \pi : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$, called its **integrated form**, which is given by

$$(u \times \pi)\left(\sum_{s \in G} a_s s\right) = \sum_{s \in G} \pi(a_s) u_s.$$

Conversely, any non-degenerate $*$-homomorphism $\rho : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$ is of the form $\rho = u \times \pi$ for some covariant representation $(u, \pi, \mathcal{H})$ [8, p. 117].

The largest C^*-norm on $C_c(G, \mathcal{A})$ that respects the group action is the full crossed product.

Definition 1.5.1. *If $\mathcal{A}$ is a unital C^*-algebra and G is a countable discrete group with a group action $\alpha : G \to Aut(\mathcal{A})$, then the* ***full crossed product*** *of $\mathcal{A}$ by G (with respect to the action α) is the completion of $C_c(G, \mathcal{A})$ with respect to the norm*

$$\|x\| = \sup \|\pi(x)\|,$$

where the supremum is taken over all Hilbert spaces $\mathcal{H}$ and all unital $*$-homomorphisms $\pi : C_c(G, \mathcal{A}) \to \mathcal{B}(\mathcal{H})$. We denote this C^*-algebra by $\mathcal{A} \rtimes_\alpha G$.

It is not hard to see that the supremum in Definition 1.5.1 is equal to the supremum taken over all non-degenerate $*$-homomorphisms. It is well-known that the norm above is faithful on $C_c(G, \mathcal{A})$; that is, if $x \in C_c(G, \mathcal{A}) \setminus \{0\}$, then $\|x\|_{\mathcal{A} \rtimes_\alpha G} \neq 0$. By construction of the full crossed product, the following universal property holds:

Proposition 1.5.2. (See [8, Proposition 4.1.3]) *Let $\alpha : G \to Aut(\mathcal{A})$ be a group action of a countable discrete group G on a unital C^*-algebra $\mathcal{A}$. If $(u, \pi, \mathcal{H})$ is a covariant representation of $C_c(G, \mathcal{A})$, then there is a $*$-homomorphism $\rho : \mathcal{A} \rtimes_\alpha G \to \mathcal{B}(\mathcal{H})$ such that*

$$\rho\left(\sum_{s \in G} a_s s\right) = \sum_{s \in G} \pi(a_s) u_s.$$

There is also a reduced crossed product associated to the algebra $C_c(G, \mathcal{A})$. To construct the reduced crossed product, we start with a faithful representation $\iota : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. We will construct what is called the **regular covariant representation** of ι as follows. We let $\ell^2(G)$ denote the Hilbert space of square-summable sequences indexed over elements of G. For $g \in G$, we let δ_g denote the function given by

$$\delta_g(h) = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

We define a representation $\pi : \mathcal{A} \to \mathcal{H} \otimes \ell^2(G)$ by setting

$$\pi(a)(v \otimes \delta_g) = (\iota(\alpha_{g^{-1}}(a)))(v) \otimes \delta_g, \quad \text{for all } v \in \mathcal{H} \text{ and } g \in G.$$

If we were to identify the Hilbert space $\mathcal{H} \otimes \ell^2(G)$ as $\bigoplus_{g \in G} \mathcal{H}$, then the representation π is like a direct sum:

$$\pi(a) = \bigoplus_{g \in G} \iota(\alpha_g^{-1}(a)).$$

The motivation for defining π in this way comes from the left regular representation of the group G. We recall that the left regular representation of G is the unitary representation $\lambda : G \to \mathcal{B}(\ell^2(G))$ given by

$$\lambda_s(\delta_g) = \delta_{sg}.$$

A key calculation (see [8, p. 117]) is that for all $s, g \in G$, $a \in \mathcal{A}$ and $v \in \mathcal{H}$,

$$(I_{\mathcal{H}} \otimes \lambda_s)\pi(a)(I_{\mathcal{H}} \otimes \lambda_s^*)(v \otimes \delta_g) = \pi(\alpha_s(a))(v \otimes \delta_g).$$

In other words, $(I_{\mathcal{H}} \otimes \lambda_s)\pi(a)(I_{\mathcal{H}} \otimes \lambda_s)^* = \pi(\alpha_s(a))$, so that the triple $(I_{\mathcal{H}} \otimes \lambda, \pi, \mathcal{H} \otimes \ell^2(G))$ is a covariant representation for $C_c(G, \mathcal{A})$.

Definition 1.5.3. *Let $\mathcal{A}$ be a unital C^*-algebra; let G be a countable discrete group; and let $\alpha : G \to Aut(\mathcal{A})$ be a group action. The* ***reduced crossed product*** *of $\mathcal{A}$ by G is the completion of $C_c(G, \mathcal{A})$ with respect to the norm given by*

$$\|x\| = \|((I_{\mathcal{H}} \otimes \lambda) \times \pi)(x)\|_{\mathcal{B}(\mathcal{H} \otimes \ell^2(G))},$$

where $(I_{\mathcal{H}} \otimes \lambda, \pi, \mathcal{H} \otimes \ell^2(G))$ is the regular covariant representation of a faithful representation $\iota : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. We denote this C^*-algebra by $\mathcal{A} \rtimes_{\alpha,r} G$.

For simplicity, for an element $x = \sum_{s \in G} a_s s$ of $C_c(G, \mathcal{A})$, we will denote by $\sum_{s \in G} a_s \lambda_s$ its image in $\mathcal{A} \rtimes_{\alpha,r} G$. It is well-known that the definition of $\mathcal{A} \rtimes_{\alpha,r} G$ does not depend on the choice of faithful representation of $\mathcal{A}$ [8, Proposition 4.1.5]. It is easy to see that, for any $x \in C_c(G, \mathcal{A})$, we have

$$\|x\|_{\mathcal{A} \rtimes_{\alpha,r} G} \leq \|x\|_{\mathcal{A} \rtimes_\alpha G}.$$

In particular, there is a canonical quotient map $q : \mathcal{A} \rtimes_\alpha G \to \mathcal{A} \rtimes_{\alpha,r} G$ that extends the identity map on $C_c(G, \mathcal{A})$.

One helpful property that the reduced crossed product possesses is a faithful conditional expectation onto $\mathcal{A}$. In other words, there is a ucp map $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,r} G \to \mathcal{A}$ with $\mathcal{E}_{|\mathcal{A}} = \mathrm{id}_{\mathcal{A}}$ and $\mathcal{E}(x) \neq 0$ for all positive elements $x \in \mathcal{A} \rtimes_{\alpha,r} G$.

Proposition 1.5.4. (See [8, Proposition 4.1.9]) *Let $\alpha : G \to Aut(\mathcal{A})$ be a group action of a countable discrete group G on a unital C^*-algebra $\mathcal{A}$. Then the map $\mathcal{E} : C_c(G, \mathcal{A}) \to \mathcal{A}$ defined by*

$$\mathcal{E}\left(\sum_{s\in G} a_s s\right) = a_e$$

extends to a faithful, conditional expectation $\mathcal{E} : \mathcal{A} \rtimes_{\alpha,r} G \to \mathcal{A}$.

A helpful observation is that, whenever $x = \sum_{s\in G} a_s s \in C_c(G, \mathcal{A})$, we have $a_s = \mathcal{E}(x\lambda_s^*)$. Moreover, $\mathcal{E}$ is G-equivariant in the sense that, for every $s \in G$ and $x \in \mathcal{A} \rtimes_{\alpha,r} G$,

$$\mathcal{E}(\lambda_s a \lambda_s^*) = \alpha_s(\mathcal{E}(x)).$$

For our purposes, the most important result regarding crossed products is the following theorem.

Theorem 1.5.5. (See [8, Theorem 4.2.6]) *Let G be a countable discrete group and let $\alpha : G \to Aut(\mathcal{A})$ be a group action of G on a unital C^*-algebra $\mathcal{A}$. If G is amenable, then the canonical quotient map $q : \mathcal{A} \rtimes_\alpha G \to \mathcal{A} \rtimes_{\alpha,r} G$ is a $*$-isomorphism.*

1.6 Free products

Many of the universal C^*-algebras that are related to Connes' embedding problem can be obtained via free products of C^*-algebras. In this section, we will introduce the main ideas and facts about free products that will be useful for us. The interested reader can also consult the work of D. Avitzour [3] and D. Voiculescu [65]. For simplicity, we will only deal with free products of unital C^*-algebras, where we identify the units of the different algebras in the free product.

For notational ease, we will consider the free product of two unital C^*-algebras; the case of an arbitrary finite collection of unital C^*-algebras follows easily. Let $\mathcal{A}_1, \mathcal{A}_2$ be unital C^*-algebras. One defines the **free product algebra $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$** (**amalgamated over the identity**) to be the algebra generated by finite words in the elements of the algebras $\mathcal{A}_i$, while identifying the units of each $\mathcal{A}_i$ with each other. Multiplication on this algebra is defined by concatenation. If adjacent letters in a word belong to the same algebra, then concatenation is interpreted as taking the product of these elements in the algebra to which they belong. For example, if $a_{1,i}, a_{2,i} \in \mathcal{A}_i$ and $a_j \in \mathcal{A}_j$ for $i \neq j$, then $(a_{1,i}) * (a_j) = a_{1,i} * a_j$ is the concatenation of the words $a_{1,i}$ and a_j, while $(a_j * a_{1,i}) * (a_{2,i}) = a_j * (a_{1,i}a_{2,i})$ is concatenation with the multiplication $a_{1,i}a_{2,i}$ being done in the algebra $\mathcal{A}_i$. The adjoint on $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is defined in a way that extends the adjoint maps on $\mathcal{A}_1$ and $\mathcal{A}_2$, while remaining an anti-$*$-homomorphism. For example, if $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, then the adjoint of $a_1 * a_2$ is $a_2^* * a_1^*$. With these definitions, $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ is a unital $*$-algebra.

The algebra $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ possesses the following universal property: if $\mathcal{C}$ is a unital C^*-algebra and $\pi_i : \mathcal{A}_i \to \mathcal{C}$ are unital $*$-homomorphisms for $i = 1, 2$, then there is a unique unital $*$-homomorphism $\pi = \pi_1 * \pi_2 : \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2 \to \mathcal{C}$ satisfying $\pi(a_i) = \pi_i(a_i)$ for all $a_i \in \mathcal{A}_i$ and $i = 1, 2$. Moreover, every unital $*$-homomorphism of $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ into a unital C^*-algebra $\mathcal{C}$ can be obtained in this manner.

To make the unital $*$-algebra $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ into a C^*-algebra, we first equip it with a norm. For $x \in \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$, we define

$$\|x\| = \sup \|\pi(x)\|_{\mathcal{C}},$$

where the supremum is taken over all unital C^*-algebras $\mathcal{C}$ and unital $*$-homomorphisms $\pi : \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2 \to \mathcal{C}$. Evidently this defines a C^*-seminorm, but it is not immediate that this is a norm. One may define

$$\mathcal{J} = \{x \in \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2 : \|x\| = 0\},$$

which is a two-sided $*$-closed algebraic ideal in $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$. Then the seminorm on $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ induces a norm on $\frac{\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2}{\mathcal{J}}$. Taking the completion of the latter space with respect to the norm yields the free product C^*-algebra of $\mathcal{A}_1$ and $\mathcal{A}_2$ amalgamated over the identity, which we will denote by $\mathcal{A}_1 * \mathcal{A}_2$. It is known that $\mathcal{J} = \{0\}$, so that the seminorm initially defined is actually a norm on $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ [3].

Related to free products is the coproduct in the category of operator systems (see [23] for more information on coproducts). If $\mathcal{S}_1$ and $\mathcal{S}_2$ are operator systems, then the **coproduct** of $\mathcal{S}_1$ and $\mathcal{S}_2$ is an operator system $\mathcal{S}_1 \oplus_1 \mathcal{S}_2$ equipped with unital, complete order embeddings $\kappa_i : \mathcal{S}_i \to \mathcal{S}_1 \oplus_1 \mathcal{S}_2$, satisfying the following universal property: whenever $\mathcal{T}$ is an operator system and $\varphi_i : \mathcal{S}_i \to \mathcal{T}$ are ucp maps for $i = 1, 2$, then there is a unique ucp map $\varphi : \mathcal{S}_1 \oplus_1 \mathcal{S}_2 \to \mathcal{T}$ such that $\varphi(\kappa_i(s_i)) = \varphi_i(s_i)$ for all $s_i \in \mathcal{S}_i$ and $i = 1, 2$. Alternatively, one can form the coproduct of $\mathcal{S}_1$ and $\mathcal{S}_2$ by considering the direct sum operator system $\mathcal{S}_1 \oplus \mathcal{S}_2$, and letting $\mathcal{J} = \text{span}\,\{(1_{\mathcal{S}_1}, -1_{\mathcal{S}_2})\}$. Then $\mathcal{S}_1 \oplus_1 \mathcal{S}_2$ is canonically isomorphic to $\frac{\mathcal{S}_1 \oplus \mathcal{S}_2}{\mathcal{J}}$ [23].

While we will be mainly interested in free products of unital C^*-algebras amalgamated over the identity as defined above (sometimes referred to as full free products), we will also use reduced free products amalgamated over the identity in Chapter 5. To define reduced free products, we require a few facts from [3].

Proposition 1.6.1. (Avitzour, [3, Propositions 1.1 and 1.4]) *Let $\mathcal{A}_1, \mathcal{A}_2$ be unital C^*-algebras, and let $\varphi_1 \in \mathcal{A}_1^*$ and $\varphi_2 \in \mathcal{A}_2^*$ be functionals with $\varphi_1(1_{\mathcal{A}_1}) = 1 = \varphi_2(1_{\mathcal{A}_2})$. Then there is a unique unital linear functional $\varphi_1 * \varphi_2 : \mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2 \to \mathbb{C}$ such that*

- $\varphi_1 * \varphi_2(a_1) = \varphi_1(a_1)$ *for all* $a_1 \in \mathcal{A}_1$,
- $\varphi_1 * \varphi_2(a_2) = \varphi_2(a_2)$ *for all* $a_2 \in \mathcal{A}_2$, *and*
- $\varphi_1 * \varphi_2(c_1 \cdots c_m) = 0$ *whenever $c_1 \cdots c_m$ is a reduced word such that each c_i belongs to either* $\ker(\varphi_1)$ *or* $\ker(\varphi_2)$, *and, for each i, we have $c_i \in \ker(\varphi_1)$ if and only if* $c_{i+1} \in \ker(\varphi_2)$.

Moreover, if φ_1 and φ_2 are tracial (i.e., $\varphi_1(a_1a_2) = \varphi_1(a_2a_1)$ for all $a_1, a_2 \in \mathcal{A}$), then $\varphi_1 * \varphi_2$ is also tracial.

The key tool for reduced free products is the GNS representation the free product state $\varphi_1 * \varphi_2$, where φ_1 and φ_2 are faithful states on $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. For this, we will construct the GNS representation of $\varphi_1 * \varphi_2$ as in [3]. For simplicity of notation, we follow the approach in [44]. We let $(\pi_1, \mathcal{H}_1, \zeta_1)$ be the GNS triple associated with φ_1. That is, $\pi_1 : \mathcal{A}_1 \to \mathcal{B}(\mathcal{H}_1)$ is a unital $*$-homomorphism and $\zeta_1 \in \mathcal{H}_1$ is a unit vector such that

$$\langle \pi_1(a_1)\zeta_1, \zeta_1 \rangle = \varphi_1(a_1) \text{ for all } a_1 \in \mathcal{A}_1.$$

Similarly, we let $(\pi_2, \mathcal{H}_2, \zeta_2)$ be the GNS triple associated with φ_2. We define $\mathcal{H}_1^0 = \{\zeta_1\}^\perp$ and $\mathcal{H}_2^0 = \{\zeta_2\}^\perp$. We define a new Hilbert space

$$\mathcal{H} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{i_j \neq i_{j+1} \text{ for all } j} \mathcal{H}_{i_1}^0 \otimes \cdots \mathcal{H}_{i_n}^0 \right)$$

We will let ζ be the unit vector in $\mathcal{H}$ with coordinate 1 in the direct summand of $\mathbb{C}$, with 0's in all other coordinates. We may identify $\mathcal{H}_i$ with $\mathbb{C}\zeta \oplus \mathcal{H}_i^0$ in $\mathcal{H}$. We will let $\mathcal{H}(i)$ be the subspace spanned by direct summands whose first tensor factor are not $\mathcal{H}_i$; in other words,

$$\mathcal{H}(i) = \mathbb{C}\zeta \oplus \bigoplus_{n \geq 1} \left(\bigoplus_{\substack{i_j \neq i_{j+1} \text{ for all } j \\ i_1 \neq i}} \mathcal{H}_{i_1}^0 \otimes \cdots \mathcal{H}_{i_n}^0 \right).$$

With these subspaces in hand, for $i = 1, 2$ we define unitaries $V_i : \mathcal{H}_i \otimes \mathcal{H}(i) \to \mathcal{H}$ by

$$\begin{aligned} V_i(\zeta_i \otimes \zeta) &= \zeta \\ V_i(h_i^0 \otimes \zeta) &= h_i^0 \text{ for all } h_i^0 \in \mathcal{H}_i^0 \\ V_i(\zeta_i \otimes (h_{i_1}^0 \otimes \cdots h_{i_n}^0) &= h_{i_1}^0 \otimes \cdots \otimes h_{i_n}^0 \\ V_i((h_i^0) \otimes (h_{i_1}^0 \otimes \cdots \otimes h_{i_n}^0)) &= h_i^0 \otimes h_{i_1}^0 \otimes \cdots \otimes h_{i_n}^0. \end{aligned}$$

We define $\rho_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$ by setting

$$\rho_i(a) = V_i(\pi_i(a) \otimes I_{\mathcal{H}(i)})V_i^*, \; \forall a \in \mathcal{A}_i.$$

Then the free product representation is given by $\pi = \rho_1 * \rho_2$.

If φ_1 and φ_2 are states, then the GNS representation of $\varphi_1 * \varphi_2$ is precisely the free product representation $\pi = \rho_1 * \rho_2$ above [3]. Therefore, the notation $\pi_{\varphi_1 * \varphi_2}$ is often used for this free product representation. If φ_1 and φ_2 are faithful, then the unit vector ζ in $\mathcal{H}$ is a separating vector for $\pi_{\varphi_1 * \varphi_2}(\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2)$, so that $\pi_{\varphi_1 * \varphi_2}$ is faithful on $\mathcal{A}_1 *_{\mathbb{C}} \mathcal{A}_2$ [3].

The fact that free products of faithful states remain faithful allow for a reduced free product corresponding to certain states. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital C^*-algebras and φ_i is a faithful state on $\mathcal{A}_i$ for each $i = 1, 2$, then the **reduced free product relative to** φ_1 and φ_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$ is given by the C^*-algebra $\pi_{\varphi_1 * \varphi_2}(\mathcal{A}_1 * \mathcal{A}_2)$. When the states φ_1, φ_2 are clear from context, we may write $\mathcal{A}_1 *_{red} \mathcal{A}_2$ for the reduced free product relative to φ_1 and φ_2.

The main example of a reduced free product we will use is $M_n *_{red} C(\mathbb{T})$, where the reduced free product is taken with respect to the canonical trace $\mathrm{tr} : M_n \to \mathbb{C}$ given by

$$\mathrm{tr}((a_{ij})) = \frac{1}{n} \sum_{i=1}^{n} a_{ii}$$

and the integration map $\tau : C(\mathbb{T}) \to \mathbb{C}$ given by

$$\tau(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\, dt.$$

We will use this construction when considering the so-called reduced Brown algebra in Chapter 5.

1.7 Probabilistic Correlation Sets

In this section, we define some of the probabilistic correlation sets. These sets arise from probability densities corresponding to finite input, finite output systems with two parties (Alice and Bob) in a separated system.

By a **projection-valued measure with k outputs**, we mean a set of mutually orthogonal projections $\{P_i\}_{i=1}^k$ on a Hilbert space $\mathcal{H}$ such that $\sum_{i=1}^k P_i = I_{\mathcal{H}}$. Let $n, k \in \mathbb{N}$ with $n, k \geq 2$. We define the set of **quantum correlations** to be

$$C_q(n,k) = \{(\langle (E_{a,x} \otimes F_{b,y})\psi, \psi\rangle)_{a,b,x,y}\} \subseteq \mathbb{R}^{n^2k^2},$$

where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert spaces; for each $1 \leq x \leq n$, Alice has a PVM $\{E_{a,x}\}_{a=1}^k$ with k outputs on $\mathcal{H}_A$; for each $1 \leq y \leq n$, Bob has a PVM $\{F_{b,y}\}_{b=1}^k$ with k outputs on $\mathcal{H}_B$, and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a unit vector.

We define $C_{qs}(n,k)$ in the same way, only dropping the requirement that $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional.

We define $C_{qc}(n,k)$ in a similar manner. The key difference is that we no longer assume a tensor product structure. Instead, we assume that each $E_{a,x}$ and $F_{b,y}$ act on the same Hilbert space $\mathcal{H}$, and that ψ is a unit vector in $\mathcal{H}$. Moreover, we assume that $E_{a,x}$ commutes with $F_{b,y}$ for each a, b, x, y.

For convenience, we define $C_{qa}(n,k)$ to be the closure of $C_q(n,k)$ in $\mathbb{R}^{n^2k^2}$.

In Chapters 5 and 6, we will need the matrix-valued generalization of these sets. For $t \in \{q, qs, qc\}$, $m, k \in \mathbb{N}$ with $m, k \geq 2$ and $n \in \mathbb{N}$, one can define the $n \times n$ matrix-valued t-correlation set in m inputs and k outputs to be the same as $C_t(m,k)$, except that, instead of having one unit vector ψ, we have a collection $\{\psi_1, ..., \psi_n\}$ of n orthonormal vectors in the Hilbert space, and the coordinates $\langle E_{a,x}F_{b,y}\psi, \psi\rangle$ are replaced by $n \times n$ matrices of the form $(\langle E_{a,x}F_{b,y}\psi_j, \psi_i\rangle)_{i,j=1}^n \in M_n(\mathbb{C})$. We then define $C_{qa}^{(n)}(m,k)$ to be the closure of $C_q^{(n)}(m,k)$ in $(M_n(\mathbb{C}))^{m^2k^2}$.

There is a connection between matrix-valued correlation sets and matrix-valued ucp maps on tensor products of a certain operator system (see [34, 25, 47] for more information). We let $\mathcal{F}_{m,k} = *_m \ell_\infty^k$, the free product of m copies of the commutative C^*-algebra ℓ_∞^k. In the algebraic tensor product $\mathcal{F}_{m,k} \otimes \mathcal{F}_{m,k}$, we will let $e_{a,x}$ be the projection onto the a-th coordinate of the x-th copy of ℓ_∞^k in the left side of the tensor product, for $1 \leq x \leq m$ and $1 \leq a \leq k$. Similarly, we will let $f_{b,y}$ be the projection onto the b-th coordinate of the y-th copy of ℓ_∞^k in the right side of the tensor product. We will set $\mathcal{S}_{m,k} = \text{span}\,\{e_{a,x} : 1 \leq x \leq m,\, 1 \leq a \leq k\} \subseteq \mathcal{F}_{m,k}$, which is an operator system.

Theorem 1.7.1. (Farenick-Kavruk-Paulsen-Todorov, [23]) *$\mathcal{S}_{m,k}$ is completely order isomorphic to the coproduct $\bigoplus_1 \{\ell_\infty^k\}_{i=1}^m$ of m copies of ℓ_∞^k.*

Taking the adjoint of the complete order isomorphism in the previous theorem yields the following corollary.

Corollary 1.7.2. (Farenick-Kavruk-Paulsen-Todorov, [23]) *The dual of $\mathcal{S}_{m,k}$ is completely order isomorphic to*

$$\mathcal{T}_{m,k} = \left\{ (\zeta_1, ..., \zeta_m) \in \bigoplus_{i=1}^m \ell_\infty^k : \sum_{j=1}^k \zeta_p(j) = \sum_{j=1}^k \zeta_q(j), \, \forall 1 \leq p, q \leq m \right\} \subseteq \bigoplus_{i=1}^m \ell_\infty^k.$$

We summarize the relations between matrix-valued correlations and states on tensor products of $\mathcal{F}_{m,k}$.

Theorem 1.7.3. *Let $m, k \geq 2$, $n \in \mathbb{N}$, and $(P(a,b|x,y))_{a,b,x,y} \in (M_n)^{m^2k^2}$. The following are equivalent:*

1. *P belongs to $C_{qc}^{(n)}(m,k)$;*
2. *There is a ucp map $\Phi : \mathcal{F}_{m,k} \otimes_{\max} \mathcal{F}_{m,k} \to M_n$ such that, for all $1 \leq x, y \leq m$ and $1 \leq a, b \leq k$,*
$$\Phi(e_{a,x} \otimes f_{b,y}) = P(a,b|x,y).$$
3. *There is a ucp map $\Phi : \mathcal{S}_{m,k} \otimes_c \mathcal{S}_{m,k} \to M_n$ such that, for all $1 \leq x, y \leq m$ and $1 \leq a, b \leq k$,*
$$\Phi(e_{a,x} \otimes f_{b,y}) = P(a,b|x,y).$$

To obtain a similar characterization for qa-correlations in terms of minimal tensor products, we require two key facts. The first fact is that commuting correlations that can be realized on a finite-dimensional Hilbert space are actually finite-dimensional tensor product correlations (Corollary 1.7.6). The second fact that we need is that, for a unital RFD C^*-algebra, the set of M_n-valued ucp maps with a finite-dimensional Stinespring representation are dense in the set of all M_n-valued ucp maps, with respect to the point-norm topology (Corollary 1.7.9). We will also use these facts when considering unitary correlations in Chapters 2 and 3.

The fact that commuting correlations on finite-dimensional space decompose as tensor product correlations relies on an observation of B. Tsirelson [63], which was generalized by V. Scholz and R. Werner [57].

Theorem 1.7.4. (Tsirelson, [63]) *Let $\mathcal{A}_A$ and $\mathcal{A}_B$ be C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$, where $\mathcal{H}$ is finite-dimensional. Suppose that $XY = YX$ for all $X \in \mathcal{A}_A$ and $Y \in \mathcal{A}_B$. Then there are finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, an isometry $V : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B$, and unital $*$-homomorphisms $\pi_A : \mathcal{A}_A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{A}_B \to \mathcal{B}(\mathcal{H}_B)$ such that, for every $X \in \mathcal{A}_A$ and $Y \in \mathcal{A}_B$, we have*

$$V^*(\pi_A(X) \otimes \pi_B(Y))V = XY.$$

The next corollary follows immediately from Theorem 1.7.4.

Corollary 1.7.5. *Let $\mathcal{A}_A$ and $\mathcal{A}_B$ be C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$, where $\mathcal{H}$ is finite-dimensional. Let $\{X_\alpha\}_\alpha \subseteq \mathcal{A}_A$ and $\{Y_\beta\}_\beta \subseteq \mathcal{A}_B$ be finite generating subsets of $\mathcal{A}_A$ and $\mathcal{A}_B$, respectively. Let $\{\eta_i\}_{i=1}^n$ be an orthonormal set of vectors in $\mathcal{H}$. If $[X_\alpha, Y_\beta] = 0$ for all α, β, then there are finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, unital $*$-homomorphisms $\pi_A : \mathcal{A}_A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{A}_B \to \mathcal{B}(\mathcal{H}_B)$, and an isometry $V : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B$ such that, for every α, β and $1 \leq i, j \leq n$, we have*

$$\langle (\pi_A(X_\alpha) \otimes \pi_B(Y_\beta))V\eta_i, V\eta_j\rangle = \langle X_\alpha Y_\beta \eta_i, \eta_j\rangle.$$

Applying the previous corollary to the context of commuting correlations, we obtain the following corollary.

Corollary 1.7.6. *Let $P = (P(a, b|x, y)) \in C_{qc}^{(n)}(m, k)$, and suppose that there is a finite-dimensional realization of P. Then $P \in C_q^{(n)}(m, k)$.*

Proof. By assumption, there are PVMs $\{e_{a,x}\}_{a=1}^k$ for each $1 \leq x \leq m$ and PVMs $\{f_{b,y}\}_{b=1}^k$ for each $1 \leq y \leq m$ on a finite-dimensional Hilbert space $\mathcal{H}$ satisfying $[e_{a,x}, f_{b,y}] = 0$ for all a, b, x, y, along with orthonormal vectors $\{\eta_i\}_{i=1}^n \subseteq \mathcal{H}$ such that

$$P(a, b|x, y) = (\langle e_{a,x} f_{b,y} \eta_j, \eta_i\rangle)_{i,j}.$$

Let $\mathcal{A}_A = C^*(\{e_{a,x} : 1 \leq x \leq m,\ 1 \leq a \leq k\})$ and $\mathcal{A}_B = C^*(\{f_{b,y} : 1 \leq y \leq m,\ 1 \leq b \leq k\})$. By Corollary 1.7.5, we may find finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ along with unital $*$-homomorphisms $\pi_A : \mathcal{A}_A \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : \mathcal{A}_B \to \mathcal{B}(\mathcal{H}_B)$ and an isometry $V : \mathcal{H} \to \mathcal{H}_A \otimes \mathcal{H}_B$ satisfying

$$\langle (\pi_A(e_{a,x}) \otimes \pi_B(f_{b,y}))V\eta_i, \eta_j\rangle = \langle e_{a,x} f_{b,y} \eta_i, \eta_j\rangle$$

for all a, x, b, y, i, j. Set $E_{a,x} = \pi_A(e_{a,x})$, $F_{b,y} = \pi_B(f_{b,y})$ and $\zeta_i = V\eta_i$. Since π_A and π_B are unital $*$-homomorphisms, each collection $\{E_{a,x}\}_{a=1}^k$ is a PVM. Similarly, each $\{F_{b,y}\}_{b=1}^k$ is a PVM. Since V is an isometry, $\langle \zeta_i, \zeta_j \rangle = \langle V\eta_i, V\eta_j \rangle = \langle V^*V\eta_i, \eta_j \rangle = \langle \eta_i, \eta_j \rangle = \delta_{ij}$, so that $\{\zeta_i\}_{i=1}^n$ is orthonormal. It readily follows that

$$P = (P(a, b|x, y)) = (\langle (E_{a,x} \otimes F_{b,y})\zeta_j, \zeta_i \rangle) \in C_q^{(n)}(m, k).$$

$\square$

Next, we need a description of unital RFD C^*-algebras in terms of M_n-valued ucp maps which have a finite-dimensional Stinespring representation. As in [21], we define $\mathrm{Fin}(\mathcal{A})$ to be the set of all states on $\mathcal{A}$ whose GNS representations act on finite-dimensional Hilbert spaces. We let $\mathcal{S}(\mathcal{A})$ be the set of all states on $\mathcal{A}$.

Theorem 1.7.7. (Exel-Loring [21]) *A unital C^*-algebra $\mathcal{A}$ is RFD if and only if Fin($\mathcal{A}$) is w^*-dense in $\mathcal{S}(\mathcal{A})$.*

For matrix-valued correlations, a generalization of the above theorem is needed. Since we only will be dealing with unital C^*-algebras, we only consider the unital case. Borrowing the notation of [21], for a unital C^*-algebra $\mathcal{A}$, we set $\mathrm{Fin}(\mathcal{A}, M_n)$ to be the set of all ucp maps $\varphi : \mathcal{A} \to M_n$ for which the minimal Stinespring representation of φ acts on a finite-dimensional Hilbert space. We will denote by $\mathrm{UCP}(\mathcal{A}, M_n)$ the set of all ucp maps from $\mathcal{A}$ into M_n. For the M_n-valued version of the theorem of Exel and Loring, we need the following standard fact.

Proposition 1.7.8. (See [49, p. 73]) *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\Phi : \mathcal{A} \to M_n$ be a ucp map. Then the map*

$$s_\Phi((a_{ij})) = \langle \Phi^{(n)}((a_{ij}))x, x \rangle,$$

where $x = \frac{1}{\sqrt{n}}(e_1 \oplus \cdots \oplus e_n)$, is a state on $M_n(\mathcal{A})$. Conversely, if $s \in \mathcal{S}(M_n(\mathcal{A}))$, then the map $\Phi_s : \mathcal{A} \to M_n$ given by

$$\Phi_s(a) = (n \cdot s(E_{ij} \otimes a))_{i,j=1}^n$$

is completely positive. The two assignments $\Phi \mapsto s_\Phi$ and $s \mapsto \Phi_s$ are mutual inverses.

We can now prove an M_n-valued version of Theorem 1.7.7.

Corollary 1.7.9. *Let $\mathcal{A}$ be a unital C^*-algebra. The following are equivalent:*

1. *$\mathcal{A}$ is RFD.*

2. *The set $\mathrm{Fin}(\mathcal{A}, M_n)$ is point-norm dense in $\mathrm{UCP}(\mathcal{A}, M_n)$ for all $n \in \mathbb{N}$.*

3. *The set $\mathrm{Fin}(\mathcal{A}, M_n)$ is point-norm dense in $\mathrm{UCP}(\mathcal{A}, M_n)$ for some $n \in \mathbb{N}$.*

Proof. Evidently (2) implies (3). If (3) holds and $x \in \mathcal{A}$, then $a := x^*x \in \mathcal{A}_+$, so that

$$\|a\| = \sup\{\|\varphi(a)\| : \varphi \in \mathrm{UCP}(\mathcal{A}, M_n)\}.$$

For $\varphi \in \mathrm{UCP}(\mathcal{A}, M_n)$, there is a net $(\varphi_\lambda)_\lambda \subseteq \mathrm{Fin}(\mathcal{A}, M_n)$ such that $\varphi(a) = \lim_\lambda \varphi_\lambda(a)$. We write $\varphi_\lambda(\cdot) = V^*\pi_\lambda(\cdot)V$ in its minimal Stinespring representation, which is finite-dimensional. Then $\|\pi_\lambda(a)\| \geq \|\varphi_\lambda(a)\|$. By the C^*-identity, we have

$$\|x\| = \sup\{\|\pi(x)\| \mid \pi : \mathcal{A} \to M_{n(\pi)} \text{ is a representation}\}.$$

This shows that $\mathcal{A}$ is RFD, so that (1) is true.

Lastly, we prove that (1) implies (2). Let $n \geq 2$, and assume that $\mathcal{A}$ is RFD. Let $\Phi : \mathcal{A} \to M_n$ be a ucp map. We use a technique similar to the proof of [15, Lemma II.5.2]. We let $x = \frac{1}{\sqrt{n}}(e_1 \oplus \cdots \oplus e_n)$. By Proposition 1.7.8, the associated map $\psi : M_n(\mathcal{A}) \to \mathbb{C}$ given by

$$\psi((a_{ij})_{i,j}) = \langle \Phi^{(n)}((a_{ij})_{i,j})x, x\rangle$$

belongs to $\mathcal{S}(M_n(\mathcal{A}))$. Since $M_n(\mathcal{A})$ is RFD, by Theorem 1.7.7 there is a net $(\psi_\lambda)_\lambda \subseteq \mathrm{Fin}(M_n(\mathcal{A}))$ such that $\lim_\lambda \psi_\lambda = \Phi$ in the point-norm topology. For each λ, we may write $\psi_\lambda(\cdot) = \langle \pi_\lambda(\cdot)\xi_\lambda, \xi_\lambda\rangle$, where $\mathcal{H}_\lambda$ is a finite-dimensional Hilbert space; $\pi_\lambda : M_n(\mathcal{A}) \to \mathcal{B}(\mathcal{H}_\lambda)$ is a unital $*$-homomorphism; and $\xi_\lambda \in \mathcal{H}_\lambda$ is a unit vector. Identifying $M_n(\mathcal{A}) = M_n \otimes \mathcal{A}$, we may apply Corollary 1.7.5 and assume without loss of generality that $\mathcal{H}_\lambda = \mathcal{K}_\lambda \otimes \mathcal{M}_\lambda$ and that $\pi_\lambda = \rho_\lambda \otimes \chi_\lambda$ for representations $\rho_\lambda : M_n \to \mathcal{B}(\mathcal{K}_\lambda)$ and $\chi_\lambda : \mathcal{A} \to \mathcal{B}(\mathcal{M}_\lambda)$. Additionally, we may assume that ξ_λ is a unit vector in $\mathcal{K}_\lambda \otimes \mathcal{M}_\lambda$.

Since ρ_λ is a unital representation of M_n, it is well-known that ρ_λ is unitarily equivalent to a direct sum of identity representations of M_n (see [15, Corollary III.1.2]). Since $\dim(\mathcal{K}_\lambda) < \infty$, there is a number $k(\lambda) \in \mathbb{N}$ and a unitary $U_\lambda : \mathcal{K}_\lambda \to \mathbb{C}^{k(\lambda)} \otimes \mathbb{C}^n$ such that $U_\lambda \rho_\lambda(X) U_\lambda^* = I_{k(\lambda)} \otimes X$ for every $X \in M_n$. We define $\zeta_\lambda = (U_\lambda \otimes I_{\mathcal{M}_\lambda})\xi$, which is a unit vector in $\mathbb{C}^{k(\lambda)} \otimes \mathbb{C}^n \otimes \mathcal{M}_\lambda$. Then for $a \in \mathcal{A}$, we have

$$\begin{aligned}
\langle (I_{k(\lambda)} \otimes E_{ij} \otimes \chi_\lambda(a))\zeta_\lambda, \zeta_\lambda\rangle &= \langle (U_\lambda \rho_\lambda(E_{ij}) U_\lambda^* \otimes \chi_\lambda(a))\zeta_\lambda, \zeta_\lambda\rangle \\
&= \langle (\rho_\lambda(E_{ij}) \otimes \chi_\lambda(a))(U_\lambda^* \otimes I_{\mathcal{M}_\lambda})\zeta_\lambda, (U_\lambda^* \otimes I_{\mathcal{M}_\lambda})\zeta_\lambda\rangle \\
&= \langle \pi_\lambda(E_{ij} \otimes a)\xi_\lambda, \xi_\lambda\rangle.
\end{aligned}$$

Now, we set $\zeta_\lambda^i = (I_{k(\lambda)} \otimes E_{ii} \otimes I_{\mathcal{M}_\lambda})\zeta_\lambda$. We may think of ζ_λ as the n-tuple $\zeta_\lambda = (\zeta_\lambda^1, ..., \zeta_\lambda^n)$, where each $\zeta_\lambda^i \in \mathbb{C}^{k(\lambda)} \otimes \mathcal{M}_\lambda$. We observe that

$$\langle (I_{k(\lambda)} \otimes E_{ij} \otimes I_{\mathcal{M}_\lambda})\zeta_\lambda, \zeta_\lambda \rangle = \langle \zeta_\lambda^j, \zeta_\lambda^i \rangle,$$

so that

$$\delta_{ij} = n\psi(E_{ij}) = \lim_\lambda n\langle (I_{k(\lambda)} \otimes E_{ij} \otimes I_{\mathcal{M}_\lambda})\zeta_\lambda, \zeta_\lambda \rangle = \lim_\lambda n\langle \zeta_\lambda^j, \zeta_\lambda^i \rangle.$$

Thus, the set of vectors $\{\sqrt{n}\zeta_\lambda^i\}_{i=1}^n$ is almost orthonormal. Define $W_\lambda : \mathbb{C}^n \to \mathbb{C}^{k(\lambda)} \otimes \mathbb{C}^n \otimes \mathcal{M}_\lambda$ by $W_\lambda e_i = \zeta_\lambda^i$. Then $\lim_\lambda \|W^*W - I\| = 0$, so eventually the polar decomposition of W_λ will be of the form $W_\lambda = V_\lambda |W_\lambda|$, where V_λ is an isometry. Moreover, $\lim_\lambda \|V_\lambda - W_\lambda\| = 0$, and

$$\begin{aligned} &\lim_\lambda |\langle (\Phi(a) - W_\lambda^*(I_{k(\lambda)} \otimes I_n \otimes \chi_\lambda(a))W_\lambda)e_j, e_i \rangle| \\ &= \lim_\lambda |n\psi(E_{ij} \otimes a) - n\langle (I_{k(\lambda)} \otimes I_n \otimes \chi_\lambda(a))\zeta_\lambda^j, \zeta_\lambda^i \rangle| \\ &= 0. \end{aligned}$$

Lastly, define a representation $\gamma_\lambda : \mathcal{A} \to \mathcal{B}(\mathbb{C}^{k(\lambda)} \otimes \mathbb{C}^n \otimes \mathcal{M}_\lambda)$ by setting

$$\gamma_\lambda(a) = I_{k(\lambda)} \otimes I_n \otimes \chi_\lambda(a).$$

Since $\lim_\lambda \|V_\lambda - W_\lambda\| = 0$, it follows that

$$\lim_\lambda \|\Phi(a) - V_\lambda^* \gamma_\lambda(a) V_\lambda\| = 0,$$

so that Φ is in the point-norm closure of $\mathrm{Fin}(\mathcal{A}, M_n)$, completing the proof. $\square$

Now we can return to the characterization of qa-correlations in terms of states on the minimal tensor product. To obtain this characterization, we note that it is well-known that the C^*-algebra $\mathcal{F}_{m,k}$ is RFD. Indeed, it is a free product (amalgamated over the identity) of a finite number of copies of a finite-dimensional algebra (see, for example, [21]). Moreover, we have seen that minimal tensor products of RFD C^*-algebras are also RFD. Applying Corollary 1.7.9 to $\mathcal{F}_{m,k} \otimes_{\min} \mathcal{F}_{m,k}$, it is not hard to obtain the following description of qa-correlations.

Theorem 1.7.10. *Let $m, k \geq 2$, $n \in \mathbb{N}$, and $(P(a,b|x,y))_{a,b,x,y} \in (M_n)^{m^2k^2}$. The following are equivalent:*

1. *P belongs to $C_{qa}^{(n)}(m,k)$;*

2. *There is a ucp map* $\Phi : \mathcal{F}_{m,k} \otimes_{\min} \mathcal{F}_{m,k} \to M_n$ *such that, for all* $1 \leq x, y \leq m$ *and* $1 \leq a, b \leq k$,
$$\Phi(e_{a,x} \otimes f_{b,y}) = P(a, b|x, y).$$
3. *There is a ucp map* $\Phi : \mathcal{S}_{m,k} \otimes_{\min} \mathcal{S}_{m,k} \to M_n$ *such that, for all* $1 \leq x, y \leq m$ *and* $1 \leq a, b \leq k$,
$$\Phi(e_{a,x} \otimes f_{b,y}) = P(a, b|x, y).$$

One can also consider correlations arising from ucp maps on the maximal tensor product of $\mathcal{S}_{m,k}$ with itself; these correlations turn out to be the set of non-signalling box correlations, as we will see below. We call an element $(P(a, b|x, y)) \in (M_n)^{m^2k^2}$ an (M_n-valued) **non-signalling box correlation** if it satisfies

- $P(a, b|x, y) \in M_n^+$ for all a, b, x, y;
- $\sum_{a,b=1}^k P(a, b|x, y) = I_n$ for all x, y;
- $\sum_{a=1}^k P(a, b|x, y) = \sum_{a=1}^k P(a, b|x', y)$ for all b, x, x', y; and
- $\sum_{b=1}^k P(a, b|x, y) = \sum_{b=1}^k P(a, b|x, y')$ for all a, x, y, y'.

The first condition is the generalization of requiring non-negative probabilities. The second condition is the generalization of requiring that the probabilities for each output pair corresponding to a fixed input pair must sum up to 1. The final two conditions are referred to as "non-signalling conditions". These conditions ensure that there are well-defined marginal probability distributions given by
$$P_A(a|x) = \sum_{b=1}^k P(a, b|x, y),$$
and
$$P_B(b|y) = \sum_{a=1}^k P(a, b|x, y).$$
These quantities represent the probability that Alice (respectively, Bob) outputs a (respectively, b), given that the input x (respectively, y) was given.

We let $C_{nsb}^{(n)}(m, k)$ be the set of M_n-valued non-signalling box correlations. In the case when $n = 1$, we simply write $C_{nsb}(m, k)$. We have the following description of non-signalling correlations.

Theorem 1.7.11. *Let $P = (P(a,b|x,y)) \in (M_n)^{m^2k^2}$. The following statements are equivalent:*

1. *P belongs to $C_{nsb}^{(n)}(m,k)$.*
2. *There is a ucp map $\Phi : \mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k} \to M_n$ such that $\Phi(e_{a,x} \otimes f_{b,y}) = P(a,b|x,y)$ for all a,b,x,y.*

Proof. We use an argument similar to the proof of [32, Theorem 2.4]. First, suppose that $\Phi : \mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k} \to M_n$ is ucp and satisfies $P(a,b|x,y) = \Phi(e_{a,x} \otimes f_{b,y})$ for all a,b,x,y. Since $e_{a,x}$ is positive in $\mathcal{S}_{m,k}$ and $f_{b,y}$ is positive in $\mathcal{S}_{m,k}$, $e_{a,x} \otimes f_{b,y}$ is positive in $\mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k}$. Therefore, $P(a,b|x,y) = \Phi(e_{a,x} \otimes f_{b,y})$ belongs to M_n^+, so that the first condition of $C_{nsb}^{(n)}(m,k)$ holds. For the second condition, we note that

$$\sum_{a,b=1}^k e_{a,x} \otimes f_{b,y} = \left(\sum_{a=1}^k e_{a,x} \right) \otimes \left(\sum_{b=1}^k f_{b,y} \right) = 1 \otimes 1.$$

Applying Φ to both sides, it follows that $\sum_{a,b=1}^k P(a,b|x,y) = I_n$, yielding the second condition. For the third condition, we observe that, in $\mathcal{S}_{m,k}$, whenever $1 \leq x, x' \leq m$, we have

$$\sum_{a=1}^k e_{a,x} = 1 = \sum_{a=1}^k e_{a,x'}.$$

Tensoring with $f_{b,y}$ and applying Φ, we obtain the third condition. Similarly, the fourth condition for $C_{nsb}^{(n)}(m,k)$ holds, so that $P \in C_{nsb}^{(n)}(m,k)$.

Conversely, suppose that $P \in C_{nsb}^{(n)}(m,k)$. We define $\Phi(e_{a,x} \otimes f_{b,y}) = P(a,b|x,y)$. By the third and fourth conditions, Φ extends to a linear functional on $\mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k}$. Since

$$\Phi(I_n) = \Phi \left(\sum_{a,b=1}^k e_{a,x} \otimes f_{b,y} \right) = \sum_{a,b=1}^k P(a,b|x,y) = I_n,$$

it follows that Φ is unital. For the last condition, we let $\Phi_{ij} : \mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k} \to \mathbb{C}$ be given by $\Phi_{ij}(x) = \Phi(x)_{ij}$, where $\Phi(x)_{ij}$ is the (i,j)-entry of the matrix $\Phi(x)$. Then Φ is ucp if and only if $F = (\Phi_{ij})_{i,j=1}^n$ is positive in $M_n((\mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k})^d)$. We note that $M_n((\mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k})^d) \simeq M_n(\mathcal{T}_{m,k} \otimes_{\min} \mathcal{T}_{m,k}) \subseteq M_n \left(\left(\bigoplus_{i=1}^m \ell_\infty^k \right) \otimes_{\min} \left(\bigoplus_{i=1}^m \ell_\infty^k \right) \right)$. Since the coordinates of F in the latter operator system are all of the form $P(a,b|x,y)$, and since each of these quantities is positive in M_n, it follows that F is positive. Hence, Φ is ucp on $\mathcal{S}_{m,k} \otimes_{\max} \mathcal{S}_{m,k}$, as desired. $\square$

1.8 Connes' Embedding problem and the Tsirelson problems

In this section, we outline some of the properties of the probabilistic correlation sets and open problems surrounding them.

We begin by examining some of the containments of the correlation sets. Since tensor product representations are automatically representations of the minimal tensor product, it is easy to see that $C_{qs}^{(n)}(m,k) \subseteq C_{qa}^{(n)}(m,k)$. On the other hand, $C_q^{(n)}(m,k) \subseteq C_{qa}^{(n)}(m,k)$ and the former set is dense in the latter. Thus, we obtain:

Corollary 1.8.1. (See [34, 25, 47]) *For each n,m,k, we have that*

$$C_q^{(n)}(m,k) \subseteq C_{qs}^{(n)}(m,k) \subseteq C_{qa}^{(n)}(m,k) \subseteq C_{qc}^{(n)}(m,k).$$

Moreover, $C_{qa}^{(n)}(m,k) = \overline{C_{qs}^{(n)}(m,k)} = \overline{C_q^{(n)}(m,k)}$.

These matrix-valued correlation sets are also convex [34, 25]. Since the matricial state space $UCP(\mathcal{A}, M_n)$ on a unital C^*-algebra $\mathcal{A}$ is compact in the point-norm topology, it is evident that $C_{qa}^{(n)}(m,k)$ and $C_{qc}^{(n)}(m,k)$ are closed sets.

In certain settings, it is easier to express matrix-valued correlations by using unitaries of order k, rather than PVMs with k outcomes. To establish this relationship, we note a well-known fact:

Proposition 1.8.2. *Let $k \in \mathbb{N}$. Let u be a generator of $\mathbb{Z}_k$; let $e_1, ..., e_k$ be the canonical basis of ℓ_∞^k, and let $\omega = \exp\left(\frac{2\pi i}{k}\right)$. There are unital $*$-homomorphisms $\pi : C^*(\mathbb{Z}_k) \to \ell_\infty^m$ and $\rho : \ell_\infty^m \to C^*(\mathbb{Z}_k)$ defined on the generators by*

$$\pi(u) = \sum_{j=1}^k \omega^j e_j$$

and

$$\rho(e_j) = \sum_{a=1}^k \omega^{-aj} u^a.$$

Moreover, π and ρ are inverses of each other, so that $C^*(\mathbb{Z}_k) \simeq \ell_\infty^k$.

Proposition 1.8.2 shows how every PVM $\{E_{a,x}\}_{a=1}^k$ with k outputs on a Hilbert space $\mathcal{H}$ can be identified with a unitary U on $\mathcal{H}$ of order k. Conversely, to each unitary of order k, there is an associated PVM with k outputs.

Using Proposition 1.8.2 and the universal property of unital free products, it is easy to see that

$$C^*(*_n\mathbb{Z}_k) \simeq *_n\ell_\infty^k,$$

for all $n, k \in \mathbb{N}$. Using this isomorphism allows one to freely pass between matrix-valued t-correlations (involving PVMs with k outputs) and the corresponding correlations obtained by replacing the PVMs with the associated unitaries of order k.

With these facts in hand, we return to the chain of inclusions from Corollary 1.8.1. Determining whether any of these inclusions are strict for $n = 1$ led to several open problems in quantum information theory:

- Is $C_q(m,k) = C_{qs}(m,k)$ for all m,k?
- (Closure problem) Is $C_{qs}(m,k) = C_{qa}(m,k)$ for all m,k? (Equivalently, is $C_{qs}(m,k)$ closed?)
- (Weak Tsirelson problem) Is $C_{qa}(m,k) = C_{qc}(m,k)$ for all m,k?
- (Strong Tsirelson problem) Is $C_{qs}(m,k) = C_{qc}(m,k)$ for all m,k?

The first problem concerns whether all tensor product correlations can be witnessed on finite-dimensional space. The Closure problem concerns whether or not limits of tensor product correlations remain tensor product correlations. The Weak Tsirelson problem concerns whether correlations from a commuting model can be arbitrarily approximated by finite-dimensional correlations. Finally, the Strong Tsirelson problem asks whether all correlations from the commuting model can be realized in a tensor product framework.

Until the last few years, all of these problems remained open. Now, there are known counterexamples to the first, second and fourth problems. The only remaining open problem is the Weak Tsirelson problem. The Strong Tsirelson problem was the first problem to be resolved negatively. Indeed, in 2016, W. Slofstra found large values of m and k for which $C_{qs}(m,k) \neq C_{qc}(m,k)$ [59]. Less than a year later, Slofstra also found m and k for which $C_{qs}(m,k) \neq C_{qa}(m,k)$, resolving the closure problem [60]. Subsequent papers [40, 20] simplified the approach to showing that $C_{qs}(m,k) \neq C_{qa}(m,k)$, while lowering the values of m and k. Due to a recent paper of K. Dykema, V. Paulsen and J. Prakash, the lowest m and k known for which $C_{qs}(m,k) \neq C_{qa}(m,k)$ is $(m,k) = (5,2)$ [20]. Determining whether $C_q(m,k) = C_{qs}(m,k)$ remained an open problem until early 2018, when A. Codalangelo and J. Stark proved that $C_q(5,3) \neq C_{qs}(5,3)$ [12]. In comparison, the Weak Tsirelson problem turns out to be equivalent to two long-standing open problems in the theory of operator algebras: Connes' embedding problem and Kirchberg's conjecture.

To state Connes' embedding problem, we require a bit of background on von Neumann algebras. We recall a theorem of F.J. Murray and J. von Neumann [46], which states that there is a unique (weakly separable) hyperfinite II_1 factor von Neumann algebra, up to isomorphism. We denote this factor by $\mathcal{R}$. There are many ways to obtain $\mathcal{R}$. For example, let $\mathcal{A} = \bigotimes_{n\in\mathbb{N}} M_2(\mathbb{C})$ be the tensor product of (countably) infinitely many copies of $M_2(\mathbb{C})$. The C^*-algebra $\mathcal{A}$, sometimes referred to as the 2^∞-UHF algebra (see, for example, [15, 8]) has a unique tracial state τ. Then $\mathcal{R} \simeq \pi_\tau(\mathcal{A})''$, where $\pi_\tau : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\tau)$ is the GNS representation of the trace (see [8, Theorem 11.5.4]). The von Neumann algebra $\mathcal{R}$ is equipped with a faithful, normal tracial state $\mathrm{tr}_{\mathcal{R}}$. Since $\mathrm{tr}_{\mathcal{R}}$ is a faithful trace, setting $\|x\|_2 = \mathrm{tr}_{\mathcal{R}}(x^*x)^{\frac{1}{2}}$ for all $x \in \mathcal{R}$ yields a norm, called the 2-norm of $\mathcal{R}$ with respect to $\mathrm{tr}_{\mathcal{R}}$.

Next, we construct the ultrapower of $\mathcal{R}$. The ultrapower construction for finite von Neumann algebras was originally done in [33, 45]. We fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. We consider the sequence space $\ell_\infty(\mathcal{R})$, the space of sequences with elements in $\mathcal{R}$, equipped with the supremum norm. We let

$$\mathcal{N}_{\mathcal{U}} = \{(x_n)_{n=1}^\infty \in \ell_\infty(\mathcal{R}) : \lim_{n\to\mathcal{U}} \|x_n\|_2 = 0\}.$$

Then the (tracial) **ultrapower** of $\mathcal{R}$ (with respect to $\mathcal{U}$) is defined by

$$\mathcal{R}^{\mathcal{U}} = \ell_\infty(\mathcal{R})/\mathcal{N}_{\mathcal{U}}.$$

It is known that $\mathcal{R}^{\mathcal{U}}$ is a von Neumann algebra, with faithful normal trace given by $\mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}((x_n)_n) = \lim_{n\to\mathcal{U}} \mathrm{tr}_{\mathcal{R}}(x_n)$.

With this terminology and background in hand, we can state Connes' embedding problem.

Problem 1.8.3. (Connes' embedding problem, [13]) *Let M be any von Neumann algebra with separable predual. Suppose that τ is a faithful, normal tracial state on M. Is there an injective, trace-preserving $*$-homomorphism $\pi : M \to \mathcal{R}^{\mathcal{U}}$?*

A seemingly unrelated problem in tensor products of C^*-algebras is the following conjecture of Kirchberg:

Conjecture 1.8.4. (Kirchberg's conjecture, [41])

$$C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty).$$

Using Theorem 1.3.6, we see that Kirchberg's conjecture is equivalent to the assertion that $C^*(\mathbb{F}_\infty)$ has the WEP. Kirchberg's conjecture is also equivalent to the analogous conjecture obtained when replacing $\mathbb{F}_\infty$ with $\mathbb{F}_n$, where $n \in \{2, 3, ...\}$. Surprisingly, Connes' embedding problem and Kirchberg's conjecture are equivalent [41]. Even more surprising is the fact that both of these problems are equivalent to Tsirelson's problem.

Theorem 1.8.5. *The following statements are equivalent.*

1. *Connes' embedding problem has a positive answer.*
2. *Kirchberg's conjecture holds.*
3. $C_{qa}^{(n)}(m,k) = C_{qc}^{(n)}(m,k)$ *for all* $m, k \geq 2$ *with* $(m,k) \neq (2,2)$ *and* $n \in \mathbb{N}$.
4. $C_{qa}(m,k) = C_{qc}(m,k)$ *for all* $m, k \geq 2$ *with* $(m,k) \neq (2,2)$.

The equivalence of (1) and (2) was obtained by E. Kirchberg [41]. The equivalence of (2) and (3) were obtained in [34] and [25]. Later, N. Ozawa showed that, in the implication (3) $\implies$ (1), it suffices to consider the case when $n = 1$, which established the equivalence of (1) and (4) [47].

We close this section with an observation on separations of $C_{t_1}^{(n)}(m,k)$ and $C_{t_2}^{(n)}(m,k)$. One of the goals in finding separations between the various correlation sets is finding minimal values of n, m and k for which a separation holds. The desire for minimality is because of the following propositions.

Proposition 1.8.6. *Let* $n_0, m_0, k_0 \in \mathbb{N}$, *and let* $n, m, k \in \mathbb{N}$ *be such that* $n \geq n_0$, $m \geq m_0$ *and* $k \geq k_0$. *If* $C_{qa}^{(n_0)}(m_0,k_0) \neq C_{qc}^{(n_0)}(m_0,k_0)$, *then* $C_{qa}^{(n)}(m,k) \neq C_{qc}^{(n)}(m,k)$.

Proof. If $C_{qa}^{(n_0)}(m_0,k_0) \neq C_{qc}^{(n_0)}(m_0,k_0)$, then there are PVMs $\{E_{a,x}\}_{a=1}^{k_0}$ and $\{F_{b,y}\}_{b=1}^{k_0}$ on $\mathcal{H}$ for $1 \leq a, b \leq m_0$, and an orthonormal set $\{\psi_1, ..., \psi_{n_0}\} \in \mathcal{H}$ such that $[E_{a,x}, F_{b,y}] = 0$ for all a, b, x, y and

$$(P(a,b|x,y))_{a,b,x,y} := ((\langle E_{a,x}F_{b,y}\psi_j, \psi_i\rangle)_{i,j=1}^{n_0})_{a,b,x,y} \in C_{qc}^{(n_0)}(m_0,k_0) \setminus C_{qa}^{(n_0)}(m_0,k_0).$$

Now we define PVMs $\{\widetilde{E}_{a,x}\}_{a=1}^k$ for $1 \leq x \leq m$ as follows. If $1 \leq x \leq m_0$, then set

$$\widetilde{E}_{a,x} = \begin{cases} E_{a,x} & 1 \leq a \leq k_0 \\ 0 & k_0 < a \leq k. \end{cases}$$

If $m_0 < x \leq m$, then we set

$$\widetilde{E}_{a,x} = \begin{cases} I_{\mathcal{H}} & a = 1 \\ 0 & a \neq 1. \end{cases}$$

We define $\widetilde{F}_{b,y}$ in a similar manner. Evidently we have $\dim(\mathcal{H}) = \infty$; otherwise we would have $P(a,b|x,y) \in C_q^{(n_0)}(m_0,k_0)$. Hence, we may extend $\{\psi_1,...,\psi_{n_0}\}$ to an orthonormal set $\{\psi_1,...,\psi_n\}$. Then

$$\widetilde{P}(a,b|x,y) := \langle \widetilde{E}_{a,x}\widetilde{F}_{b,y}\psi_j,\psi_i\rangle_{i,j=1}^n$$

defines an element of $C_{qc}^{(n)}(m,k)$. Moreover, $\widetilde{P}(a,b|x,y) = 0$ whenever $a > k_0$ or $b > k_0$.

If we had $\widetilde{P} \in C_{qa}^{(n)}(m,k)$, then since $C_{qa}^{(n)}(m,k) = \overline{C_{qs}^{(n)}(m,k)}$, there would be a sequence of correlations $(\widetilde{P}_\ell(a,b|x,y)) \in C_{qs}^{(n)}(m,k)$ such that

$$\lim_{\ell\to\infty} \widetilde{P}_\ell(a,b|x,y) = \widetilde{P}(a,b|x,y),\ \forall a,b,x,y.$$

Then for each $\ell \in \mathbb{N}$, there are Hilbert spaces $\mathcal{H}_\ell$ and $\mathcal{K}_\ell$, projections $\widetilde{E}_{\ell,a,x}$ on $\mathcal{H}_\ell$ and $\widetilde{F}_{\ell,b,y}$ on $\mathcal{K}_\ell$ respectively, and an orthonormal set $\{\psi_{\ell,1},...,\psi_{\ell,n}\} \in \mathcal{H}_\ell \otimes \mathcal{K}_\ell$ such that $\sum_{a=1}^k \widetilde{E}_{\ell,a,x} = I_{\mathcal{H}_\ell}$ for all x, $\sum_{b=1}^k \widetilde{F}_{\ell,b,y} = I_{\mathcal{K}_\ell}$ for all y, and

$$\widetilde{P}_\ell(a,b|x,y) = (\langle(\widetilde{E}_{\ell,a,x} \otimes \widetilde{F}_{\ell,b,y})\psi_{\ell,j},\psi_{\ell,i}\rangle)_{i,j=1}^n.$$

Next, we define m_0 PVMs on $\mathcal{H}_\ell$ with k_0 outputs by setting, for $1 \leq x \leq m_0$,

$$E_{\ell,a,x} = \begin{cases} \widetilde{E}_{\ell,a,x} & 1 \leq a \leq k_0 - 1 \\ I_{\mathcal{H}_\ell} - \sum_{p=0}^{k_0-1} \widetilde{E}_{\ell,p,x} & a = k_0. \end{cases}$$

We define $F_{\ell,b,y}$ on $\mathcal{K}_\ell$ for $1 \leq b \leq m_0$ and $1 \leq y \leq k_0$ in a similar manner. Then the element

$$(P_\ell(a,b|x,y))_{\substack{1\leq x,y\leq m_0\\ 1\leq a,b\leq k_0}} = ((\langle(E_{\ell,a,x} \otimes F_{\ell,b,y})\psi_{\ell,j},\psi_{\ell,i}\rangle)_{i,j=1}^{n_0})_{\substack{1\leq x,y\leq m_0\\ 1\leq a,b\leq k_0}}$$

belongs to $C_{qs}^{(n_0)}(m_0,k_0)$ for all ℓ. Evidently $\lim_{\ell\to\infty} P_\ell(a,b|x,y) = P(a,b|x,y)$ whenever $a,b \leq k_0$. Since $P_\ell(a,b|x,y) = \widetilde{P}_\ell(a,b|x,y)$ whenever $a \leq k_0$ and $b \leq k_0$, and since

$\widetilde{P}_\ell(a,b|x,y) \to 0$ if $a > k_0$ or $b > k_0$, we have $\sum_{a,b=1}^{k_0} P_\ell(a,b|x,y) \to I_n$. This forces $\lim_{\ell\to\infty} P_\ell(a,b|x,y) = 0 = P(a,b|x,y)$ whenever $a > k_0$ or $b > k_0$. In other words, P is the pointwise limit of $(P_\ell)_{\ell=1}^\infty$, so that P must be in $C_{qa}^{(n_0)}(m_0,k_0)$, which is a contradiction. Hence, $C_{qa}^{(n)}(m,k) \neq C_{qc}^{(n)}(m,k)$. □

A similar result holds for separations between the qs and qa models.

Proposition 1.8.7. *Let $n \geq n_0$, $m \geq m_0$, and $k \geq k_0$. If $C_{qs}^{(n_0)}(m_0,k_0) \neq C_{qa}^{(n_0)}(m_0,k_0)$, then $C_{qs}^{(n)}(m,k) \neq C_{qa}^{(n)}(m,k)$.*

Proof. Choose $(P(a,b|x,y)) \in C_{qa}^{(n_0)}(m_0,k_0)$. For each $\ell \in \mathbb{N}$, we may choose Hilbert spaces $\mathcal{H}_\ell$, $\mathcal{K}_\ell$, PVMs $\{E_{\ell,a,x}\}_{a=1}^{k_0}$ on $\mathcal{H}_\ell$ and $\{F_{\ell,b,y}\}_{b=1}^{k_0}$ on $\mathcal{K}_\ell$ for $1 \leq a,b \leq m_0$, and an orthonormal set $\{\psi_{\ell,1},...,\psi_{\ell,n_0}\} \in \mathcal{H}_\ell \otimes \mathcal{K}_\ell$ such that

$$P(a,b|x,y) = \lim_{\ell\to\infty} P_\ell(a,b|x,y),$$

where

$$P_\ell(a,b|x,y) := (\langle (E_{\ell,a,x} \otimes F_{\ell,b,y})\psi_{\ell,j},\psi_{\ell,i}\rangle)_{i,j=1}^{n_0}.$$

If $1 \leq x \leq m_0$, then set

$$\widetilde{E}_{\ell,a,x} = \begin{cases} E_{\ell,a,x} & 1 \leq a \leq k_0 \\ 0 & k_0 < a \leq k. \end{cases}$$

If $m_0 < x \leq m$, then we set

$$\widetilde{E}_{\ell,a,x} = \begin{cases} I_{\mathcal{H}} & a = 1 \\ 0 & a \neq 1. \end{cases}$$

We define $\widetilde{F}_{\ell,b,y}$ in a similar manner. Then for each ℓ and x, $\{\widetilde{E}_{\ell,a,x}\}_{x=1}^k$ is a PVM on $\mathcal{H}_\ell$. Similarly, for each ℓ and y, $\{\widetilde{F}_{\ell,b,y}\}_{y=1}^k$ is a PVM on $\mathcal{K}_\ell$. If $\sup_{\ell\in\mathbb{N}} \dim(\mathcal{H}_\ell) = p < \infty$ and $\sup_{\ell\in\mathbb{N}} \dim(\mathcal{K}_\ell) = q < \infty$, then by passing to a subsequence if necessary, we could find limits $E_{a,x} = \lim_{\ell\to\infty} E_{\ell,a,x}$ and $F_{b,y} = \lim_{\ell\to\infty} F_{\ell,b,y}$, along with limits $\psi_i = \lim_{\ell\to\infty} \psi_{\ell,i}$. Then each $\{E_{a,x}\}_{a=1}^{k_0}$ is a PVM on a Hilbert space $\mathcal{H} \simeq \mathbb{C}^p$, and each $\{F_{b,y}\}_{b=1}^{k_0}$ is a PVM on a Hilbert space $\mathcal{K} \simeq \mathbb{C}^q$, while $P(a,b|x,y) = (\langle (E_{a,x} \otimes F_{b,y})\psi_j,\psi_i\rangle)_{i,j=1}^n$, so that $P \in C_{qs}^{(n_0)}(m_0,k_0)$, which is a contradiction. Hence, $\sup_{\ell\in\mathbb{N}} \dim(\mathcal{H}_\ell \otimes \mathcal{K}_\ell) = \infty$. In particular, we may assume that ℓ is large enough so that $n \leq \dim(\mathcal{H}_\ell \otimes \mathcal{K}_\ell)$. Then we extend $\{\psi_{\ell,1},...,\psi_{\ell,n_0}\}$ to an orthonormal set $\{\psi_{\ell,1},...,\psi_{\ell,n}\}$. Then

$$\widetilde{P}_\ell(a,b|x,y) := (\langle \widetilde{E}_{\ell,a,x}\widetilde{F}_{\ell,b,y}\psi_{\ell,j},\psi_{\ell,i}\rangle)_{i,j=1}^n$$

defines an element of $C_{qs}^{(n)}(m,k)$ for each ℓ. Since $C_{qa}^{(n)}(m,k)$ is compact in $(M_n)^{m^2k^2}$, by passing to a subsequence if necessary, we may assume that $\lim_{\ell\to\infty}\widetilde{P}_\ell(a,b|x,y) = P(a,b|x,y)$ for all a,b,x,y. Then $\widetilde{P}$ is an element of $C_{qa}^{(n)}(m,k)$. Moreover, $\widetilde{P}(a,b|x,y)=0$ whenever $a>k_0$ or $b>k_0$.

If we had $\widetilde{P}\in C_{qs}^{(n)}(m,k)$, then there would be Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, along with PVMs $\{\widetilde{E}_{a,x}\}_{x=1}^k$ on $\mathcal{H}$ and PVMs $\{\widetilde{F}_{b,y}\}_{y=1}^k$ on $\mathcal{K}$ for $1\le a,b\le m$, along with an orthonormal set $\{\psi_1,...,\psi_n\}\subseteq\mathcal{H}\otimes\mathcal{K}$ such that

$$\widetilde{P}(a,b|x,y)=(\langle(\widetilde{E}_{a,x}\otimes\widetilde{F}_{b,y})\psi_j,\psi_i\rangle)_{i,j=1}^n.$$

Now we define m_0 PVMs on $\mathcal{H}$ with k_0 outputs by setting, for $1\le x\le m_0$,

$$E_{a,x}=\begin{cases}\widetilde{E}_{a,x} & 1\le a\le k_0-1\\ I_{\mathcal{H}}-\sum_{p=0}^{k_0-1}\widetilde{E}_{p,x} & a=k_0.\end{cases}$$

We define $F_{b,y}$ on $\mathcal{K}$ for $1\le b\le m_0$ and $1\le y\le k_0$ in a similar manner. Then the element

$$(Q(a,b|x,y))_{\substack{1\le x,y\le m_0\\1\le a,b\le k_0}}=((\langle(E_{a,x}\otimes F_{b,y})\psi_j,\psi_i\rangle)_{i,j=1}^{n_0})_{\substack{1\le x,y\le m_0\\1\le a,b\le k_0}}$$

belongs to $C_{qs}^{(n_0)}(m_0,k_0)$. Evidently $Q(a,b|x,y)=P(a,b|x,y)$ whenever $a,b\le k_0$. Since $\widetilde{P}(a,b|x,y)=0$ if $a>k_0$ or $b>k_0$, it follows that $P(a,b|x,y)=Q(a,b|x,y)$ when $a=k_0$ or $b=k_0$. In other words, $P=Q\in C_{qs}^{(n_0)}(m_0,k_0)$, which is a contradiction. Thus, $C_{qs}^{(n_0)}(m_0,k_0)\neq C_{qa}^{(n_0)}(m_0,k_0)$. ☐

Similar to the above propositions is the case of $t_1=q$ and $t_2=qs$. The proof is largely the same as the previous proposition, and is omitted.

Proposition 1.8.8. *Let $n\ge n_0$, $m\ge m_0$ and $k\ge k_0$. If $C_q^{(n_0)}(m_0,k_0)\neq C_{qs}^{(n_0)}(m_0,k_0)$, then $C_q^{(n)}(m,k)\neq C_{qs}^{(n)}(m,k)$.*

1.9 Synchronous Correlations

In this section, we will outline the main characterizations of synchronous correlations in each of the q, qs, qa and qc models. See [19, 40, 50, 51] for more information on synchronous correlations. Our motivation for this outline is that it yields an alternative proof that Connes' embedding problem is equivalent to determining whether $C_{qa}(m,2)=C_{qc}(m,2)$

for all $m \geq 2$. This equivalence is one of the assertions in Ozawa's theorem [47, Theorem 36], but the proof given in [47] relies on facts about linear spans of projections in II_1-factors. Alternatively, with the recent resolution of the description of the synchronous qa correlations [40], one can give a direct proof that $C_{qa}^s(n,k) = C_{qc}^s(n,k)$ if and only if $C_{qa}^s(nk,2) = C_{qc}^s(nk,2)$ (see Lemma 1.9.4). Applying this fact yields Ozawa's condition.

Definition 1.9.1. *Let $(p(a,b|x,y)) \in C_{nsb}(n,k)$. We say that p is* ***synchronous*** *if $p(a,b|x,x) = 0$ whenever $a \neq b$.*

For each $t \in \{q, qs, qa, qc, nsb\}$, we write $C_t^s(n,k)$ for the subset of $C_t(n,k)$ consisting of synchronous correlations. In [50], the following descriptions were given for $C_q^s(n,k)$ and $C_{qc}^s(n,k)$.

Theorem 1.9.2. (See [50, Corollary 5.6]) *Let $(p(a,b|x,y)) \in \mathbb{R}^{n^2k^2}$. Then:*

1. *The correlation p belongs to $C_{qc}^s(n,k)$ if and only if there is a unital C^*-algebra $\mathcal{A}$ generated by projections $\{e_{a,x} : 1 \leq x \leq n,\ 1 \leq a \leq k\}$ such that $\sum_{a=1}^k e_{a,x} = 1$ for all x, along with a tracial state $\tau : \mathcal{A} \to \mathbb{C}$ satisfying*

$$p(a,b|x,y) = \tau(e_{a,x}e_{b,y}),\ \forall a,b,x,y.$$

2. *The correlation p belongs to $C_q^s(n,k)$ if and only if there are projections $E_{a,x} \in M_N(\mathbb{C})$ for some $N \in \mathbb{N}$ satisfying $\sum_{a=1}^k E_{a,x} = I_N$ for each x and*

$$p(a,b|x,y) = \mathrm{tr}(E_{a,x}E_{b,y}),\ \forall a,b,x,y.$$

The description of the synchronous qa correlations was left as an open problem. Related to this problem was determining whether $C_{qa}^s(n,k)$ is the closure of $C_q^s(n,k)$. The subtlety lies in the fact that, if $(p(a,b|x,y)) \in C_{qa}^s(n,k)$, then there are correlations $(p_\ell(a,b|x,y)) \in C_q(n,k)$ with $\lim_{\ell\to\infty} p_\ell(a,b|x,y) = p(a,b|x,y)$ for each a,b,x,y, but we may not have $p_\ell(a,b|x,x) = 0$ for $a \neq b$. These problems were resolved by S.-J. Kim, V. Paulsen and C. Schafhauser.

Theorem 1.9.3. (Kim-Paulsen-Schafhauser, [40, Theorem 3.6]) *Fix a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and let $n,k \geq 2$. Let $(p(a,b|x,y)) \in \mathbb{R}^{n^2k^2}$. The following are equivalent:*

1. *$(p(a,b|x,y)) \in C_{qa}^s(n,k)$.*

2. *There is a sequence* $((p_\ell(a,b|x,y)))_{\ell=1}^\infty \in C_q^s(n,k)$ *such that*

$$\lim_{\ell\to\infty} p_\ell(a,b|x,y) = p(a,b|x,y) \text{ for all } a,b,x,y.$$

3. *There are projections* $e_{a,x} \in \mathcal{R}^\mathcal{U}$ *such that* $\sum_{a=1}^k e_{a,x} = 1$ *for all* $1 \le x \le n$ *and*

$$\mathrm{tr}_{\mathcal{R}^\mathcal{U}}(e_{a,x}e_{b,y}) = p(a,b|x,y),\ \forall a,b,x,y,$$

where $\mathcal{R}^\mathcal{U}$ *is the tracial ultrapower of the hyperfinite* II_1 *factor* $\mathcal{R}$.

In particular, $C_{qa}^s(n,k)$ *is the closure of* $C_q^s(n,k)$.

Using the above description of $C_{qa}^s(n,k)$, we obtain the following lemma, which provides a useful transformation between $C_t^s(n,k)$ and $C_t^s(nk,2)$. We thank Chris Schafhauser for suggesting this transformation.

Lemma 1.9.4. *Suppose that* $p(a,b|x,y) = \tau(E_{a,x}E_{b,y}) \in C_{qc}^s(n,k)$, *where* τ *is a trace on a unital* C^**-algebra* $\mathcal{A}$ *and each* $E_{a,x}$ *is a projection in* $\mathcal{A}$ *satisfying* $\sum_{a=1}^k E_{a,x} = 1_\mathcal{A}$ *for all* $1 \le x \le n$. *For* $i = 0,1$, *define*

$$\widetilde{E}_{i,(a,x)} = \begin{cases} E_{a,x} & i = 1 \\ I - E_{a,x} & i = 0. \end{cases} \tag{1.9.1}$$

Then $\widetilde{p}(i,j|(a,x),(b,y)) := \tau(\widetilde{E}_{i,(a,x)}\widetilde{E}_{j,(b,y)})$ *defines an element of* $C_{qc}^{(s)}(nk,2)$. *Moreover, if* $\widetilde{p} \in C_{qa}^s(nk,2)$, *then* $p \in C_{qa}^s(n,k)$.

Proof. The only part of the lemma that is not immediate is that $p \in C_{qa}^s(n,k)$ if $\widetilde{p} \in C_{qa}^s(nk,2)$. If we have $\widetilde{p} \in C_{qa}^s(nk,2)$, then by Theorem 1.9.3, there are projections $e_{a,x} \in \mathcal{R}^\mathcal{U}$ such that, for all a,b,x,y,

$$p(a,b|x,y) = \widetilde{p}(1,1|(a,x),(b,y)) = \mathrm{tr}_{\mathcal{R}^\mathcal{U}}(e_{a,x}e_{b,y}). \tag{1.9.2}$$

Since $\mathrm{tr}_{\mathcal{R}^\mathcal{U}}$ is a trace, we also have $\mathrm{tr}_{\mathcal{R}^\mathcal{U}}(e_{a,x}e_{b,y}) = \mathrm{tr}_{\mathcal{R}^\mathcal{U}}(e_{b,y}e_{a,x}e_{b,y})$. Since p is synchronous, for each $a \neq b$, we have

$$\mathrm{tr}_{\mathcal{R}^\mathcal{U}}(e_{b,x}e_{a,x}e_{b,x}) = p(a,b|x,x) = 0. \tag{1.9.3}$$

Since $\mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}$ is faithful and $e_{b,x}e_{a,x}e_{b,x} = (e_{b,x}e_{a,x})^*(e_{a,x}e_{b,x})$, we have $e_{a,x}e_{b,x} = 0$. Thus, $e_{a,x} \perp e_{b,x}$ for $a \neq b$, so that $\sum_{a=1}^{k} e_{a,x}$ is a projection in $\mathcal{R}^{\mathcal{U}}$. By definition of p, for each x,

$$\sum_{a,b=1}^{k} \mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}(e_{b,x}e_{a,x}e_{b,x}) = \sum_{a=1}^{k} \mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}(e_{a,x}) = \sum_{a=1}^{k} p(a,a|x,x) = 1. \tag{1.9.4}$$

Therefore, $\mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}\left(1 - \sum_{a=1}^{k} e_{a,x}\right) = 0$. Since $1 - \sum_{a=1}^{k} e_{a,x} \geq 0$, faithfulness of $\mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}$ forces $\sum_{a=1}^{k} e_{a,x} = 1$. In particular, for each $1 \leq x \leq n$, $\{e_{a,x}\}_{a=1}^{k}$ is a PVM with k outcomes. Since $p(a,b|x,y) = \mathrm{tr}_{\mathcal{R}^{\mathcal{U}}}(e_{a,x}e_{b,y})$ for all a,b,x,y, by Theorem 1.9.3 it follows that $p \in C_{qa}^{s}(n,k)$. □

Using Lemma 1.9.4, we obtain:

Theorem 1.9.5. *The following are equivalent.*

1. *Connes' embedding problem has a positive answer.*
2. $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.
3. $C_{qa}(n,k) = C_{qc}(n,k)$ *for all* $n,k \geq 2$.
4. $C_{qa}(m,2) = C_{qc}(m,2)$ *for all* $m \geq 2$.
5. $C_{qa}^{s}(n,k) = C_{qc}^{s}(n,k)$ *for all* $n,k \geq 2$.
6. $C_{qa}^{s}(m,2) = C_{qc}^{s}(m,2)$ *for all* $m \geq 2$.

Proof. The equivalence of (1), (2) and (3) is the content of Theorem 1.8.5. Clearly (3) implies (4), which implies (6). The previous lemma shows that (5) and (6) are equivalent. By a result of K. Dykema and V. Paulsen [19, Proposition 3.2], having $\overline{C_q^s(n,k)} = C_{qc}^s(n,k)$ for all n,k implies that Connes' embedding problem has a positive answer. Combining this implication with the fact that $C_{qa}^s(n,k) = \overline{C_q^s(n,k)}$, we see that (5) implies (1). □

Chapter 2

A non-commutative unitary analogue of Kirchberg's conjecture

In this chapter, we will consider the Brown algebra $\mathcal{U}_{nc}(n)$ and the operator system $\mathcal{V}_n$ spanned by the generators of $\mathcal{U}_{nc}(n)$. In Section 2.1, we will show that $\mathcal{V}_n$ is an operator system quotient of M_{2n}, and that $\mathcal{V}_n$ and $\mathcal{U}_{nc}(n)$ have the local lifting property. In Section 2.2, we will use these facts to reformulate Connes' embedding problem in terms of a version of Kirchberg's conjecture obtained when replacing $C^*(\mathbb{F}_\infty)$ with $\mathcal{U}_{nc}(n)$. On the way, we will obtain an alternate proof of Kirchberg's theorem that $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{B}(\mathcal{H}) = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{B}(\mathcal{H})$. Finally, in Section 2.3 we will show that the embedding problem is equivalent to having $\mathcal{V}_n \otimes_{\min} \mathcal{V}_n$ and $\mathcal{V}_n \otimes_c \mathcal{V}_n$ order isomorphic for each $n \geq 2$. The latter problem allows us to consider problems involving $\mathcal{U}_{nc}(n)$ in terms of states on tensor products of $\mathcal{V}_n$, which we will use as a basis for studying unitary correlation sets in Chapter 3.

2.1 The Brown algebra $\mathcal{U}_{nc}(n)$

For any $n \in \mathbb{N}$, we denote by $\mathcal{U}_{nc}(n)$ the universal C^*-algebra on n^2 generators $\{u_{ij}\}_{i,j=1}^n$ with the restriction that the matrix $U = (u_{ij})$ is unitary. This algebra is sometimes referred to as the **Brown algebra**, and it was first defined by L. Brown in [7]. It has the universal property that, whenever $\mathcal{A}$ is a unital C^*-algebra and $a_{ij} \in \mathcal{A}$ are such that (a_{ij}) is unitary in $M_n(\mathcal{A})$, then there is a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \to \mathcal{A}$ such that $\pi(u_{ij}) = a_{ij}$ for all $1 \leq i, j \leq n$. We may define the operator subsystem

$$\mathcal{V}_n = \text{span } (\{1\} \cup \{u_{ij}\}_{i,j=1}^n \cup \{u_{ij}^*\}_{i,j=1}^n).$$

We sometimes refer to $\mathcal{V}_n$ as the **Brown operator system**. This operator system possesses an important universal property.

Proposition 2.1.1. *Let $T = (T_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ be a contraction. Then there is a unique ucp map $\psi : \mathcal{V}_n \to \mathcal{B}(\mathcal{H})$ such that $\psi(u_{ij}) = T_{ij}$ for all $1 \leq i, j \leq n$. The ucp map ψ dilates to a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \to M_2(\mathcal{B}(\mathcal{H}))$ such that*

$$\pi(u_{ij}) = \begin{pmatrix} T_{ij} & (\sqrt{I - TT^*})_{ij} \\ (\sqrt{I - T^*T})_{ij} & -T_{ji}^* \end{pmatrix}$$

for all $1 \leq i, j \leq n$.

Proof. Let $T = (T_{ij})$; then $\|T\| \leq 1$. The operator $V = \begin{pmatrix} T & \sqrt{I - TT^*} \\ \sqrt{I - T^*T} & -T^* \end{pmatrix}$ is unitary in $M_2(M_n(\mathcal{B}(\mathcal{H})))$. Performing a canonical shuffle (see [49, p. 97]) yields a unitary $W = (W_{ij}) \in M_n(M_2(\mathcal{B}(\mathcal{H})))$ such that

$$W_{ij} = \begin{pmatrix} T_{ij} & (\sqrt{I - TT^*})_{ij} \\ (\sqrt{I - T^*T})_{ij} & -T_{ji}^* \end{pmatrix}.$$

By the universal property of $\mathcal{U}_{nc}(n)$, there is a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \to M_2(\mathcal{B}(\mathcal{H}))$ with $\pi(u_{ij}) = W_{ij}$ for all i, j. Compressing to the $(1,1)$-corner in $M_2(\mathcal{B}(\mathcal{H}))$ yields the ucp map ψ, as desired. $\square$

We remark that $\mathcal{V}_n$ is the image of a unital, completely positive map on M_{2n}. Indeed, define $\varphi : M_{2n} \to \mathcal{V}_n$ by

$$\varphi(E_{ij}) = \begin{cases} \frac{1}{2n} \cdot 1 \text{ if } i = j \\ \frac{1}{2n} u_{i,j-n} \text{ if } i \leq n \text{ and } j \geq n+1 \\ \frac{1}{2n} u_{j,i-n}^* \text{ if } i \geq n+1 \text{ and } j \leq n \\ 0 \text{ otherwise.} \end{cases}$$

Then the Choi matrix of φ is $(\varphi(E_{ij})) = \begin{pmatrix} \frac{1}{2n} I & \frac{1}{2n} U \\ \frac{1}{2n} U^* & \frac{1}{2n} I \end{pmatrix}$. Since $U^*U = I$, $(\varphi(E_{ij}))$ is positive in $M_2(M_n(\mathcal{V}_n))_+$, so that φ is unital and completely positive by a theorem of Choi (see [49, Theorem 3.14]). We will denote by $\mathcal{J}_{2n}$ the kernel of φ. Then

$$\mathcal{J}_{2n} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M_2(M_n) : \text{Tr}(A \oplus B) = 0 \right\}.$$

It is readily checked that $\mathcal{J}_{2n}$ contains no positive or negative elements except 0. It follows by Proposition 1.1.3 that $\mathcal{J}_{2n}$ is completely order proximinal.

To simplify notation, we define for each $n \geq 2$ the index sets

$$\Lambda_n^+ = \{(i,j) \in \{1,...,2n\}^2 : i \leq n,\, j \geq n+1\}$$

and

$$\Lambda_n^- = \{(i,j) \in \{1,...,2n\}^2 : i \geq n+1,\, j \leq n\}.$$

We define $\Lambda_n = \Lambda_n^+ \cup \Lambda_n^-$. Using the notation of [24], whenever $i,j \in \{1,...,2n\}$ are such that $\varphi(E_{ij}) \neq 0$, we define $e_{ij} = \dot{E}_{ij} \in M_{2n}/\mathcal{J}_{2n}$. We let $q : M_{2n} \to M_{2n}/\mathcal{J}_{2n}$ be the canonical quotient map. To show that $\mathcal{V}_n$ is a complete quotient of M_{2n}, we need some equivalent characterizations of positivity in the quotient operator system $M_{2n}/\mathcal{J}_{2n}$. This result and its proof are analogous to [24, Proposition 2.3]. The key difference is the use of the universal property of $\mathcal{V}_n$ in the proof that (6) implies (5) below.

Lemma 2.1.2. *Let $A_{11}, A_{ij} \in M_p$ for every $(i,j) \in \Lambda_n$. The following are equivalent.*

1. *$\dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes A_{ij}$ is positive in $(M_{2n}/\mathcal{J}_{2n}) \otimes M_p$.*
2. *$\dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} \dot{\psi}(e_{ij}) \otimes A_{ij}$ is positive in $M_r \otimes M_p$ whenever $r \in \mathbb{N}$ and $\dot{\psi} : M_{2n}/\mathcal{J}_{2n} \to M_r$ is ucp.*
3. *$\dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} \psi(E_{ij}) \otimes A_{ij}$ is positive in $M_r \otimes M_p$ whenever $r \in \mathbb{N}$ and $\psi : M_{2n} \to M_r$ is ucp with $\psi(\mathcal{J}_{2n}) = \{0\}$.*
4. *Whenever $B_{ij} \in M_r$ for $(i,j) \in \Lambda_n^+$ and the matrix $B = (B_{i,j+n}) \in M_n(M_r)$ are such that*
$$\begin{pmatrix} \frac{1}{2n} I_r & B \\ B^* & \frac{1}{2n} I_r \end{pmatrix}$$
is positive in $M_2(M_n(M_r))$, then
$$I_r \otimes A_{11} + \sum_{(i,j)\in\Lambda_n^+} B_{ij} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} B_{ji}^* \otimes A_{ij} \in (M_r \otimes M_p)_+.$$
5. *Whenever $C_{ij} \in M_r$ for $(i,j) \in \Lambda_n^+$ and the matrix $C = (C_{i,j+n}) \in M_n(M_r)$ is such that*
$$\begin{pmatrix} I_r & C \\ C^* & I_r \end{pmatrix}$$

is positive in $M_2(M_n(M_r))$, *then*

$$I_r \otimes A_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} C_{ij} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} C_{ji}^* \otimes A_{ij} \in (M_r \otimes M_p)_+.$$

6. $I_r \otimes A_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes A_{ij}$ *is positive in* $\mathcal{V}_n \otimes M_p$.

7. $I_r \otimes (2nA_{11}) + \sum_{i=1}^{2n} E_{ii} \otimes B_i + \sum_{(i,j)\in\Lambda_n} E_{ij} \otimes A_{ij}$ *is positive in* $M_{2n} \otimes M_p$ *for some matrices* $B_1, ..., B_{2n} \in M_p$ *such that* $\sum_{i=1}^{2n} B_i = (2n - 4n^2)A_{11}$.

8. $\sum_{i=1}^{2n} E_{ii} \otimes R_{ii} + \sum_{(i,j)\in\Lambda_n} E_{ij} \otimes R_{ij}$ *is positive in* $M_{2n} \otimes M_p$ *for some matrix* $R = (R_{ij}) \in (M_{2n} \otimes M_p)_+$ *such that* $R_{ij} = A_{ij}$ *for* $(i,j) \in \Lambda_n$ *and* $\sum_{i=1}^{2n} R_{ii} = 2nA_{11}$.

Proof. Suppose that (1) holds. Since $\mathcal{J}_{2n}$ is completely order proximinal, there exists a matrix $R = \sum_{i,j=1}^{2n} E_{ij} \otimes R_{ij} \in (M_{2n} \otimes M_p)_+$ such that $q \otimes \mathrm{id}_p(R) = \dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes A_{ij}$. It follows that

$$\dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes A_{ij} = q \otimes \mathrm{id}_p \left(\sum_{i,j=1}^{2n} E_{ij} \otimes R_{ij} \right) = \sum_{i=1}^{2n} \frac{1}{2n} e_{ii} \otimes R_{ii} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes R_{ij}.$$

Therefore, $R_{ij} = A_{ij}$ whenever $(i,j) \in \Lambda_n$, and $\sum_{i=1}^{2n} R_{ii} = 2nA_{11}$, which shows that (8) is true.
Assume that (8) is true, and let $R = (R_{ij}) \in (M_{2n} \otimes M_p)_+$ be as given. Write $R_{ii} = 2nA_{11} + B_i$ where $B_i = -\sum_{j\neq i} R_{jj}$. Then

$$2nA_{11} = \sum_{i=1}^{2n} R_{ii} = 4n^2 A_{11} + \sum_{i=1}^{2n} B_i.$$

Hence, $\sum_{i=1}^{2n} B_i = (2n - 4n^2)A_{11}$ and (7) follows.
If (7) is true, then applying $\varphi \otimes \mathrm{id}_p$ shows that

$$1 \otimes (2nA_{11}) + \sum_{i=1}^{2n} \frac{1}{2n} 1 \otimes B_i + \sum_{(i,j)\in\Lambda_n^+} u_{i,j-n} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} u_{j,i-n}^* \otimes A_{ij} \in (\mathcal{V}_n \otimes M_p)_+.$$

Using the fact that $\sum_{i=1}^{2n} \frac{1}{2n} 1 \otimes B_i = \frac{1}{2n} 1 \otimes \left(\sum_{i=1}^{2n} B_i\right) = (1-2n) 1 \otimes A_{11}$ shows that

$$1 \otimes A_{11} + \sum_{(i,j)\in\Lambda_n^+} u_{i,j-n} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} u_{j,i-n}^* \otimes A_{ij} \in (\mathcal{V}_n \otimes M_p)_+.$$

Thus, (7) implies (6).

Suppose that (6) is true, and let $C_{ij} \in M_r$ for $(i,j) \in \Lambda_n^+$ be such that $\begin{pmatrix} I_r & C \\ C^* & I_r \end{pmatrix} \in (M_2(M_n(M_r)))_+$, where $C = (C_{i,j+n}) \in M_n(M_r)$. Then C is a contraction in $M_n(M_r)$. By Proposition 2.1.1, there is a ucp map $\psi : \mathcal{V}_n \to M_r$ such that $\psi(u_{ij}) = C_{i,j+n}$ for all $1 \leq i, j \leq n$. Applying $\psi \otimes \mathrm{id}_p$ to the positive element in (6) shows that

$$I_r \otimes A_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} C_{ij} \otimes A_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} C_{ji}^* \otimes A_{ij} \in (M_r \otimes M_p)_+.$$

Therefore, (5) is true.

Assume (5). Let $B_{ij} \in M_r$ for $(i,j) \in \Lambda_n^+$ and $B = (B_{i,j+n}) \in M_n(M_r)$ be such that $\begin{pmatrix} \frac{1}{2n} I_r & B \\ B^* & \frac{1}{2n} I_r \end{pmatrix}$ is in $(M_2(M_n(M_r)))_+$. Let $C_{ij} = 2nB_{ij}$, so that $B = \frac{1}{2n}C$, where $C = (C_{i,j+n})$. Then (5) immediately implies (4).

Suppose that (4) holds, and let $\psi : M_{2n} \to M_r$ be a ucp map such that $\psi(\mathcal{J}_{2n}) = \{0\}$. Since $E_{ii} - E_{jj} \in \mathcal{J}_{2n}$ for all $i \neq j$, we have $\psi(E_{ii}) = \frac{1}{2n} I_r$ for all $1 \leq i \leq 2n$. If $B_{ij} = \psi(E_{ij})$ for $(i,j) \in \Lambda_n^+$, then the Choi matrix of ψ is $\begin{pmatrix} \frac{1}{2n} I_r & B \\ B^* & \frac{1}{2n} I_r \end{pmatrix}$, and hence must be positive. Then (3) follows from (4).

If (3) is true and $\dot{\psi} : M_{2n}/\mathcal{J}_{2n} \to M_r$ is ucp, then $\psi := \dot{\psi} \circ q : M_{2n} \to M_r$ is ucp and annihilates $\mathcal{J}_{2n}$. This shows that (2) holds.

Finally, suppose that (2) is true. Let $h = \dot{1} \otimes A_{11} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes A_{ij}$. Note that an element x of $(M_{2n}/\mathcal{J}_{2n}) \otimes M_p$ is positive if and only if whenever $r \in \mathbb{N}$ and $\gamma : (M_{2n}/\mathcal{J}_{2n}) \otimes M_p \to M_r$ is ucp, then $\gamma(x) \in (M_r)_+$ (see [10]). Now, if $\gamma : (M_{2n}/\mathcal{J}_{2n}) \otimes M_p \to M_r$ is ucp, then we may find linear maps $V_1, ..., V_m : \mathbb{C}^p \to \mathbb{C}^r \otimes \mathbb{C}^p$ and ucp maps $\dot{\psi}_1, ..., \dot{\psi}_m : M_{2n}/\mathcal{J}_{2n} \to M_r$ such that

$$\gamma = \sum_{i=1}^{m} V_i^* (\dot{\psi}_i \otimes \mathrm{id}_p(\cdot)) V_i.$$

Applying (2), we see that $\gamma(h) \in (M_r)_+$ for each ucp map $\gamma : (M_{2n}/\mathcal{J}_{2n}) \otimes M_p \to M_r$ and for each $r \in \mathbb{N}$. Thus, h is positive in $(M_{2n}/\mathcal{J}_{2n}) \otimes M_p$. Therefore, (1) follows from (2), as desired. □

Theorem 2.1.3. *For $\varphi : M_{2n} \to \mathcal{V}_n$ and for $\mathcal{J}_{2n}$ as above, the following are true:*

1. *The map $\varphi : M_{2n} \to \mathcal{V}_n$ is a complete quotient map; i.e., $M_{2n}/\mathcal{J}_{2n}$ is completely order isomorphic to $\mathcal{V}_n$.*

2. The C^*-envelope of $\mathcal{V}_n$ is $\mathcal{U}_{nc}(n)$.

Proof. The proof is similar to the proof of [24, Theorem 2.4]. Since φ is a surjection, the map $\dot{\varphi} : M_{2n}/\mathcal{J}_{2n} \to \mathcal{V}_n$ given by $\dot{\varphi}(\dot{x}) = \varphi(x)$ is ucp and a linear bijection. Using the fact that statements (1) and (6) are equivalent in Lemma 2.1.2, we see that $\dot{\varphi}$ is a complete order isomorphism, which proves the first statement.

For the second statement, we will show that $\mathcal{U}_{nc}(n)$ satisfies the universal property of $C_e^*(\mathcal{V}_n)$. Let $\mathcal{A}$ be any unital C^*-algebra equipped with a unital complete order embedding $\iota : \mathcal{V}_n \to \mathcal{A}$ such that $C^*(\iota(\mathcal{V}_n)) = \mathcal{A}$. We assume that $\mathcal{U}_{nc}(n)$ is represented faithfully on some Hilbert space $\mathcal{H}$. The identity map $\mathrm{id} : \mathcal{V}_n \to \mathcal{V}_n \subseteq \mathcal{U}_{nc}(n)$ can be written as $\mathrm{id} = \kappa \circ \iota$, where $\kappa : \iota(\mathcal{V}_n) \to \mathcal{V}_n$ is the ucp inverse of ι. We extend κ to a ucp map $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by Arveson's extension theorem [2]. Let $\rho = V^*\pi(\cdot)V$ be a minimal Stinespring representation of ρ on some Hilbert space $\mathcal{H}_\pi = \mathrm{ran}(V) \oplus \mathrm{ran}(V)^\perp$. With respect to this decomposition, for all $1 \leq i, j \leq n$, we have

$$\pi(\iota(u_{ij})) = \begin{pmatrix} u_{ij} & * \\ * & * \end{pmatrix}.$$

Let $U = (u_{ij})$. The matrix $\pi^{(n)} \circ \iota^{(n)}(U) = (\pi \circ \iota(u_{ij}))_{i,j=1}^n$, after applying the canonical shuffle, looks like

$$\begin{pmatrix} U & * \\ * & * \end{pmatrix}.$$

Since U is unitary and $\pi \circ \iota$ is completely contractive, the $(1, 2)$ and $(2, 1)$ blocks must be 0. By applying the inverse shuffle, it follows that for all i, j, we have

$$\pi(\iota(u_{ij})) = \begin{pmatrix} u_{ij} & 0 \\ 0 & * \end{pmatrix}.$$

Thus, ρ is multiplicative on the generators $\{\iota(u_{ij})\}_{i,j=1}^n$ of $\mathcal{A}$, so that ρ is a $*$-homomorphism with $\rho(\iota(u_{ij})) = u_{ij}$ for all i, j. This shows that ρ is surjective from $\mathcal{A}$ onto $\mathcal{U}_{nc}(n)$. By the universal property of C^*-envelopes, we conclude that $C_e^*(\mathcal{V}_n) = \mathcal{U}_{nc}(n)$. □

Using the fact that $M_{2n}/\mathcal{J}_{2n} \simeq \mathcal{V}_n$ allows for a description of the dual of $\mathcal{V}_n$. For an operator space $X \subseteq \mathcal{B}(\mathcal{H})$, we define the operator system

$$\mathcal{S}_X^0 = \left\{ \begin{pmatrix} \lambda I_{\mathcal{H}} & x \\ y^* & \lambda I_{\mathcal{H}} \end{pmatrix} : \lambda \in \mathbb{C},\, x, y \in X \right\}.$$

Corollary 2.1.4. *The operator system dual $\mathcal{V}_n^d$ of $\mathcal{V}_n$ is completely order isomorphic to $\mathcal{S}_{M_n}^0$.*

Proof. We use the same argument as in [24, Proposition 2.7]. Since $\varphi : M_{2n} \to \mathcal{V}_n$ is a complete quotient map, $\varphi^d : \mathcal{V}_n^d \to M_{2n}^d$ is a complete order embedding by Proposition 1.1.5. If $\{\delta_{ij}\}_{i,j=1}^{2n}$ is the dual basis for M_{2n}^d of the canonical basis $\{E_{ij}\}_{i,j=1}^{2n}$ for M_{2n}, then M_{2n} is completely order isomorphic to M_{2n}^d via the mapping $E_{ij} \mapsto \delta_{ij}$ by Theorem 1.1.2. It follows that the vector space dual of $M_{2n}/\mathcal{J}_{2n}$, equipped with the operator system structure inherited from M_{2n}, is the operator system dual of $\mathcal{V}_n$. It is not hard to see that the vector space dual of $M_{2n}/\mathcal{J}_{2n}$ is the annihilator of $\mathcal{J}_{2n}$ in M_{2n}^d. Therefore, $\mathcal{V}_n^d \simeq \mathcal{S}_{M_n}^0$. $\square$

We will now move towards an analogue of Kirchberg's Theorem for $\mathcal{U}_{nc}(n)$. Kirchberg's famous result on the full group C^*-algebra of the free group $\mathbb{F}_n$, for $n \geq 2$, is that $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{B}(\mathcal{H}) = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{B}(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. We will show that a similar result is true when replacing $C^*(\mathbb{F}_n)$ by $\mathcal{U}_{nc}(n)$.

First, we adopt some terminology using Lemma 2.1.2. We say that an operator system $\mathcal{S}$ has **property** $\mathfrak{V}_n$ if whenever $p \in \mathbb{N}$ and $S_{11}, S_{ij} \in M_p(\mathcal{S})$ for $(i,j) \in \Lambda_n$ are such that

$$1 \otimes S_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes S_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes S_{ij} \in (\mathcal{U}_{nc}(n) \otimes_{\min} M_p(\mathcal{S}))_+,$$

then for each $\varepsilon > 0$ there exist $R_{ij}^\varepsilon \in M_p(\mathcal{S})$ for $1 \leq i,j \leq 2n$ such that

- The matrix $R_\varepsilon = (R_{ij}^\varepsilon)$ is positive in $M_{2n}(M_p(\mathcal{S}))$;
- $R_{ij}^\varepsilon = S_{ij}$ for all $(i,j) \in \Lambda_n$; and
- $\sum_{i=1}^{2n} R_{ii}^\varepsilon = 2n(S_{11} + \varepsilon 1_{M_p(\mathcal{S})})$.

Equivalently, $\mathcal{S}$ has property $\mathfrak{V}_n$ if and only if the above holds when replacing the above positive element of $\mathcal{U}_{nc}(n) \otimes_{\min} M_p(\mathcal{S})$ with

$$\dot{1} \otimes S_{11} + \sum_{(i,j)\in\Lambda_n} e_{ij} \otimes S_{ij} \in (M_{2n}/\mathcal{J}_{2n} \otimes_{\min} M_p(\mathcal{S}))_+.$$

We will say that $\mathcal{S}$ has **property** $\mathfrak{V}$ if it has property $\mathfrak{V}_n$ for every $n \in \mathbb{N}$. These properties were inspired by the similar notion of operator systems having property $\mathfrak{W}_{n+1}$ with regards to the operator system $\mathcal{W}_{n+1} \subseteq C^*(\mathbb{F}_n)$ given by $\mathcal{W}_{n+1} = \text{span}\,\{w_i w_j^* : 1 \leq i,j \leq n+1\}$, where $w_2, ..., w_{n+1}$ are the generators of F_n and $w_1 = 1$ (see [24]). Lemma 2.1.2 shows that M_p has property $\mathfrak{V}$ for every $p \in \mathbb{N}$. The operator systems satisfying property $\mathfrak{V}_n$ are characterized in the following proposition.

Proposition 2.1.5. *Let $\mathcal{S}$ be an operator system. Then $\mathcal{S}$ has property $\mathfrak{V}_n$ if and only if $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_{\max} \mathcal{S}$. In particular, if $\mathcal{S}$ is* $(\min, \max)$*-nuclear, then $\mathcal{S}$ has property $\mathfrak{V}$.*

Proof. We proceed as in the proof of [24, Proposition 3.3]. Let $X = (x_{k\ell}) \in M_r(\mathcal{V}_n \otimes \mathcal{S})$ for $r > 1$; then

$$x_{k\ell} = 1 \otimes s_{11}^{(k\ell)} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes s_{ij}^{(k\ell)} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes s_{ij}^{(k\ell)},$$

so if $S_{11} = (s_{11}^{(k\ell)})$ and $S_{ij} = (s_{ij}^{(k\ell)})$, then we obtain

$$X = 1 \otimes S_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes S_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes S_{ij},$$

where $S_{11}, S_{ij} \in M_r$. Now, $\mathcal{S}$ satisfies the definition of property $\mathfrak{V}_n$ when $p = r$ if and only if $M_r(\mathcal{S})$ satisfies property $\mathfrak{V}_n$ when $p = 1$. Moreover, we have $M_r(\mathcal{V}_n \otimes_{\min} \mathcal{S}) = \mathcal{V}_n \otimes_{\min} M_r(\mathcal{S})$ and $M_r(\mathcal{V}_n \otimes_{\max} \mathcal{S}) = \mathcal{V}_n \otimes_{\max} M_r(\mathcal{S})$. By replacing $\mathcal{S}$ with $M_r(\mathcal{S})$ if necessary, in order to check that $\mathcal{S}$ has property $\mathfrak{V}_n$, it suffices to show that $\mathcal{S}$ satisfies the definition of property $\mathfrak{V}_n$ when $p = 1$.

Suppose that $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_{\max} \mathcal{S}$, and suppose that

$$x := 1 \otimes s_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes s_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes s_{ij} \in (\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S})_+.$$

Then $x \in (\mathcal{V}_n \otimes_{\max} \mathcal{S})_+$. Hence, for every $\varepsilon > 0$, $x + \varepsilon(1 \otimes 1_{\mathcal{S}}) \in D_1^{\max}(\mathcal{V}_n, \mathcal{S})$. This means that there is $V \in M_k(\mathcal{V}_n)_+$, $S \in M_m(\mathcal{S})_+$ and a linear map $A : \mathbb{C}^k \otimes \mathbb{C}^m \to \mathbb{C}$ such that

$$x + \varepsilon(1 \otimes 1_{\mathcal{S}}) = A(V \otimes S)A^*.$$

Since $\varphi : M_{2n} \to \mathcal{V}_n$ is a complete quotient map, there is $R \in M_k(M_{2n})_+$ such that

$$x + \varepsilon(1 \otimes 1_{\mathcal{S}}) = A(\varphi(R) \otimes S)A^*.$$

So, with $R_\varepsilon = A(R \otimes S)A^* \in M_{2n}(\mathcal{S})_+$, we have $\varphi \otimes \mathrm{id}_{\mathcal{S}}(R_\varepsilon) = x + \varepsilon(1 \otimes 1_{\mathcal{S}})$. That is to say, for each $\varepsilon > 0$, there is $R_\varepsilon \in M_{2n}(\mathcal{S})_+$ such that

$$x + \varepsilon(1 \otimes 1_{\mathcal{S}}) = \sum_{i=1}^{2n} 1 \otimes R_{ii}^\varepsilon + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes R_{ij}^\varepsilon + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes R_{ij}^\varepsilon.$$

Comparing coefficients with the coefficients of $x + \varepsilon(1 \otimes 1_{\mathcal{S}})$ shows that $R^{\varepsilon}_{ij} = s_{ij}$ for $(i,j) \in \Lambda_n$ and $\frac{1}{2n}\sum_{i=1}^{2n} R^{\varepsilon}_{ii} = s_{11} + \varepsilon 1$. This shows that $\mathcal{S}$ has property $\mathfrak{V}_n$.

Conversely, suppose that $\mathcal{S}$ has property $\mathfrak{V}_n$ and let $p \in \mathbb{N}$; we must show that $\mathcal{C}_p^{\min}(\mathcal{V}_n, \mathcal{S}) \subseteq \mathcal{C}_p^{\max}(\mathcal{V}_n, \mathcal{S})$. As before, by replacing $\mathcal{S}$ with $M_r(\mathcal{S})$ if necessary, we may assume that $p = 1$. Let $x \in (\mathcal{V}_n \otimes_{\min} \mathcal{S})_+$. Then there are $s_{11}, s_{ij} \in \mathcal{S}$ for $(i,j) \in \Lambda_n$ such that

$$x = 1 \otimes s_{11} + \sum_{(i,j)\in\Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes s_{ij} + \sum_{(i,j)\in\Lambda_n^-} \frac{1}{2n} u^*_{j,i-n} \otimes s_{ij}.$$

Since $\mathcal{S}$ has property $\mathfrak{V}_n$, given $\varepsilon > 0$, there are $R^{\varepsilon}_{ij} \in \mathcal{S}$ for $1 \leq i,j \leq 2n$ such that $R_\varepsilon = (R^{\varepsilon}_{ij}) \in M_{2n}(\mathcal{S})_+$, $R^{\varepsilon}_{ij} = s_{ij}$ for all $(i,j) \in \Lambda_n$ and $\sum_{i=1}^{2n} R^{\varepsilon}_{ii} = 2n(s_{11} + \varepsilon 1_{\mathcal{S}})$. If $\varphi : M_{2n} \to \mathcal{V}_n$ is the complete quotient map given as before, then the map $\varphi \otimes \mathrm{id}_{\mathcal{S}} : M_{2n} \otimes_{\max} \mathcal{S} \to \mathcal{V}_n \otimes_{\max} \mathcal{S}$ is ucp, and

$$\varphi \otimes \mathrm{id}_{\mathcal{S}}(R_\varepsilon) = x + \varepsilon(1 \otimes 1_{\mathcal{S}}) \in (\mathcal{V}_n \otimes_{\max} \mathcal{S})_+.$$

Thus, $x + \varepsilon(1 \otimes 1_{\mathcal{S}}) \in \mathcal{D}_1^{\max}(\mathcal{V}_n, \mathcal{S})$ for all $\varepsilon > 0$. Therefore, $x \in (\mathcal{V}_n \otimes_{\max} \mathcal{S})_+$, which completes the proof. $\square$

The next fact about tensor products of $\mathcal{V}_n$ is very useful.

Proposition 2.1.6. *Let $\mathcal{S}$ be any operator system. For all $n \geq 2$, the inclusion $\mathcal{V}_n \otimes_c \mathcal{S} \subseteq \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$ is a complete order embedding.*

Proof. By Theorem 1.2.17, $\mathcal{A} \otimes_c \mathcal{S} = \mathcal{A} \otimes_{\max} \mathcal{S}$ for every unital C^*-algebra $\mathcal{A}$. We must show that $C_p^{\mathrm{comm}}(\mathcal{V}_n, \mathcal{S}) = C_p^{\mathrm{comm}}(\mathcal{U}_{nc}(n), \mathcal{S}) \cap M_p(\mathcal{V}_n \otimes \mathcal{S})$ for each $p \in \mathbb{N}$. The inclusion map $\iota_n : \mathcal{V}_n \to \mathcal{U}_{nc}(n)$ is ucp, so by functoriality of the commuting tensor product, $\iota_n \otimes \mathrm{id}_{\mathcal{S}} : \mathcal{V}_n \otimes_c \mathcal{S} \to \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$ is ucp. Therefore, $C_p^{\mathrm{comm}}(\mathcal{V}_n, \mathcal{S}) \subseteq C_p^{\mathrm{comm}}(\mathcal{U}_{nc}(n), \mathcal{S}) \cap M_p(\mathcal{V}_n \otimes \mathcal{S})$ for each p.

Conversely, suppose that $X \in C_p^{\mathrm{comm}}(\mathcal{U}_{nc}(n), \mathcal{S}) \cap M_p(\mathcal{V}_n \otimes \mathcal{S})$. Let $\psi : \mathcal{V}_n \to \mathcal{B}(\mathcal{H})$ and $\gamma : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be ucp maps with commuting ranges. Let $T = (T_{ij})$ where $T_{ij} = \psi(u_{ij})$. By Proposition 2.1.1, the map ψ dilates to a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \to M_2(\mathcal{B}(\mathcal{H}))$ such that, for all $1 \leq i,j \leq n$,

$$\pi(u_{ij}) = \begin{pmatrix} T_{ij} & (\sqrt{I - TT^*})_{ij} \\ (\sqrt{I - T^*T})_{ij} & -T^*_{ji} \end{pmatrix}.$$

We extend ψ to a ucp map on all of $\mathcal{U}_{nc}(n)$ by letting $\psi(x)$ be given by the $(1,1)$ corner of $\pi(x)$, for each $x \in \mathcal{U}_{nc}(n)$. Define $\widetilde{\gamma} : \mathcal{S} \to M_2(\mathcal{B}(\mathcal{H}))$ by setting

$$\widetilde{\gamma}(s) = \begin{pmatrix} \gamma(s) & 0 \\ 0 & \gamma(s) \end{pmatrix}.$$

Since $\gamma(s)$ commutes with each T_{ij} and T_{ij}^*, we see that $\gamma(s) \otimes I_n$ commutes with T and T^*. Hence, $\gamma(s) \otimes I_n$ commutes with $C^*(I_{\mathcal{H}}, T, T^*)$, which contains $\sqrt{I - T^*T}$ and $\sqrt{I - TT^*}$. Thus, $\gamma(s)$ commutes with each block of $\pi(u_{ij})$. It follows that the range of $\widetilde{\gamma}$ commutes with each $\pi(u_{ij})$. Since π is a $*$-homomorphism, the ucp maps π and $\widetilde{\gamma}$ must have commuting ranges. Therefore, the map $\pi \cdot \widetilde{\gamma} : \mathcal{U}_{nc}(n) \otimes_c \mathcal{S} \to M_2(\mathcal{B}(\mathcal{H}))$ is ucp. Compressing to the $(1,1)$ corner in $M_2(\mathcal{B}(\mathcal{H}))$ yields the map $\psi \cdot \gamma$. It follows that $(\psi \cdot \gamma)_{|\mathcal{V}_n \otimes \mathcal{S}}$ is ucp on the inclusion of $\mathcal{V}_n \otimes \mathcal{S}$ into $\mathcal{U}_{nc}(n) \otimes_c \mathcal{S}$. Hence, $(\psi \cdot \gamma)^{(p)}(X) \in M_p(\mathcal{B}(\mathcal{H}))_+$. As ψ and γ were arbitrary, we conclude that $X \in C_p^{\text{comm}}(\mathcal{V}_n, \mathcal{S})$. $\square$

Lemma 2.1.7. *Let $\mathcal{S}$ be any operator system. Then $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_c \mathcal{S}$ if and only if $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$. In particular, if $\mathcal{S}$ has property $\mathfrak{V}_n$, then $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$.*

Proof. Suppose that $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$. Since the min tensor product is injective, $\mathcal{V}_n \otimes_{\min} \mathcal{S}$ is completely order isomorphic to the image of $\mathcal{V}_n \otimes \mathcal{S}$ in $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S}$. By Proposition 2.1.6, $\mathcal{V}_n \otimes_c \mathcal{S}$ is completely order isomorphic to its image in $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$. It follows that $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_c \mathcal{S}$.

Conversely, suppose that $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_c \mathcal{S}$. We employ an argument analogous to the proof of [24, Proposition 3.6]. Let $X \in M_p(\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S})_+$, and let $\psi : \mathcal{U}_{nc}(n) \to \mathcal{B}(\mathcal{H})$ and $\gamma : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ be ucp maps with commuting ranges. Let $\psi = V^* \pi(\cdot) V$ be a minimal Stinespring representation for ψ on some Hilbert space $\mathcal{H}_\pi$. By Arveson's commutant lifting theorem [2, Theorem 1.3.1], there is a unital $*$-homomorphism $\rho : (\psi(\mathcal{U}_{nc}(n)))' \to (\pi(\mathcal{U}_{nc}(n)))'$ such that $\rho(T)V = VT$ for all $T \in (\psi(\mathcal{U}_{nc}(n)))'$. Since ψ and γ have commuting ranges, we see that $\widetilde{\gamma} = \rho \circ \gamma : \mathcal{S} \to (\pi(\mathcal{U}_{nc}(n)))' \subseteq \mathcal{B}(\mathcal{K})$ is ucp and its range commutes with the range of π. Since $\mathcal{V}_n \otimes_{\min} \mathcal{S} = \mathcal{V}_n \otimes_c \mathcal{S}$, the map $(\pi \cdot \widetilde{\gamma})_{|\mathcal{V}_n \otimes_{\min} \mathcal{S}}$ is ucp.

Since the min tensor product is injective, $\mathcal{V}_n \otimes_{\min} \mathcal{S}$ is completely order isomorphic to the image of $\mathcal{V}_n \otimes \mathcal{S}$ in $\mathcal{U}_{nc}(n) \otimes_{\min} C_e^*(\mathcal{S})$. Arveson's extension theorem [2] guarantees existence of a ucp extension $\eta : \mathcal{U}_{nc}(n) \otimes_{\min} C_e^*(\mathcal{S}) \to \mathcal{B}(\mathcal{K})$ of $(\pi \cdot \widetilde{\gamma})_{|\mathcal{V}_n \otimes_{\min} \mathcal{S}}$. For any $1 \leq i, j \leq n$, we see that

$$\eta(u_{ij} \otimes 1) = \pi \cdot \widetilde{\gamma}(u_{ij} \otimes 1) = \pi(u_{ij}).$$

Thus, $\{u_{ij} : 1 \leq i, j \leq n\}$ is in the multiplicative domain $\mathcal{M}_\eta$ of η. It follows that $\mathcal{U}_{nc}(n) \otimes 1 \subseteq \mathcal{M}_\eta$. Hence whenever $a \in \mathcal{U}_{nc}(n)$ and $s \in \mathcal{S}$, we obtain

$$\eta(a \otimes s) = \eta(\underbrace{(a \otimes 1)}_{\in \mathcal{M}_\eta}(1 \otimes s)) = \eta(a \otimes 1)\eta(1 \otimes s) = \pi(a)\widetilde{\gamma}(s).$$

Now, the upper-left corner of $\pi(a)\widetilde{\gamma}(s)$ is

$$V^*\pi(a)\widetilde{\gamma}(s)V = V^*\pi(a)\rho(\gamma(s))V = V^*\pi(a)V\gamma(s) = \psi(a)\gamma(s).$$

Using this fact, we have

$$\psi \cdot \gamma(a \otimes s) = \psi(a)\gamma(s) = V^*\pi(a)\widetilde{\gamma}(s)V = V^*\eta(a \otimes s)V.$$

So, for all $z \in \mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S}$ we have $\psi \cdot \gamma(z) = V^*\eta(z)V$, so that $(\psi \cdot \gamma)_{|\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S}}$ is ucp. Therefore, $(\psi \cdot \gamma)^{(n)}(X) \in M_p(M_m)_+$ so that $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} = \mathcal{U}_{nc}(n) \otimes_c \mathcal{S}$.

$\square$

Lemma 2.1.8. *If $\mathcal{H}$ is any Hilbert space, then $\mathcal{B}(\mathcal{H})$ has property $\mathfrak{V}$. Equivalently, $\mathcal{V}_n$ has the OSLLP for every $n \geq 2$.*

Proof. The proof proceeds in a similar manner to the proof of [24, Proposition 3.5]. Let $n \in \mathbb{N}$ and let $S_{11}, S_{ij} \in \mathcal{B}(\mathcal{H})$ for $(i,j) \in \Lambda_n$. Suppose that

$$1 \otimes S_{11} + \sum_{(i,j) \in \Lambda_n^+} \frac{1}{2n} u_{i,j-n} \otimes S_{ij} + \sum_{(i,j) \in \Lambda_n^-} \frac{1}{2n} u_{j,i-n}^* \otimes S_{ij} \in (\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{B}(\mathcal{H}))_+.$$

As in the proof of Proposition 2.1.5, we may assume that $p = 1$. The matrix $0 \in M_n$ is a contraction, so by Proposition 2.1.1, the map $\alpha : \mathcal{V}_n \to \mathbb{C}$ given by $\alpha(u_{ij}) = 0$ and $\alpha(1) = 1$ extends to a ucp map on $\mathcal{U}_{nc}(n)$. Hence, $\alpha \otimes \mathrm{id}_{\mathcal{B}(\mathcal{H})}$ is ucp on $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{B}(\mathcal{H})$, which forces $S_{11} \geq 0$. Fix $\varepsilon > 0$. For any finite-dimensional subspace $\mathcal{M}$ of $\mathcal{H}$, we know that $\mathcal{B}(\mathcal{M})$ has property $\mathfrak{V}$. Let $P_\mathcal{M}$ denote the orthogonal projection onto $\mathcal{M}$. Replacing S_{11} with $P_\mathcal{M} S_{11} P_\mathcal{M}$ and S_{ij} with $P_\mathcal{M} S_{ij} P_\mathcal{M}$, we may find $R_{ij}^{\varepsilon,\mathcal{M}} \in \mathcal{B}(\mathcal{M})$ such that

- $R_{\varepsilon,\mathcal{M}} := (R_{ij}^{\varepsilon,\mathcal{M}})$ is in $(M_{2n}(\mathcal{B}(\mathcal{M})))_+$,
- $R_{ij}^{\varepsilon,\mathcal{M}} = P_\mathcal{M} S_{ij} P_\mathcal{M}$ for $(i,j) \in \Lambda_n$, and
- $\sum_{i=1}^{2n} R_{ii}^{\varepsilon,\mathcal{M}} = 2n(P_\mathcal{M} S_{11} P_\mathcal{M} + \varepsilon I_\mathcal{M}) = 2n(P_\mathcal{M} S_{11} P_\mathcal{M} + \varepsilon P_\mathcal{M})$.

Clearly $\sum_{i=1}^{2n} R_{ii}^{\varepsilon,\mathcal{M}} \leq 2n(S_{11} + \varepsilon I_{\mathcal{H}})$, so since each $R_{ii}^{\varepsilon,\mathcal{M}}$ is positive, the diagonal blocks of $R_{\varepsilon,\mathcal{M}}$ are bounded. Since $\mathcal{M}$ is finite-dimensional and $R_{\varepsilon,\mathcal{M}} \geq 0$, the norm of $R_{\varepsilon,\mathcal{M}}$ is given by the largest eigenvalue. Therefore, indexing finite-dimensional subspaces of $\mathcal{H}$ by inclusion, the net $(R_{\varepsilon,\mathcal{M}})_{\mathcal{M}\leq\mathcal{H},\,\dim(\mathcal{M})<\infty}$ is uniformly bounded. Let R_ε be a w^*-limit point of the net $(R_{\varepsilon,\mathcal{M}})_{\mathcal{M}}$. Then the corresponding subnet of $(P_{\mathcal{M}})_{\mathcal{M}}$ converges strongly to $I_{\mathcal{H}}$. It follows that, writing $R_\varepsilon = (R_{ij}^\varepsilon) \in M_n(\mathcal{B}(\mathcal{H}))$, we have $R_\varepsilon \geq 0$, while $R_{ij}^\varepsilon = S_{ij}$ for all $(i,j) \in \Lambda_n$ and $\sum_{i=1}^{2n} R_{ii}^\varepsilon = 2n(S_{11} + \varepsilon I_{\mathcal{H}})$. Therefore, $\mathcal{B}(\mathcal{H})$ has property $\mathfrak{V}_n$ for every $n \in \mathbb{N}$. $\square$

Theorem 2.1.9. *$\mathcal{U}_{nc}(n)$ has the LLP; that is, $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{B}(\mathcal{H}) = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{B}(\mathcal{H})$.*

Proof. By Lemma 2.1.8 and Proposition 2.1.5, $\mathcal{V}_n \otimes_{\min} \mathcal{B}(\mathcal{H}) = \mathcal{V}_n \otimes_{\max} \mathcal{B}(\mathcal{H})$. Applying Lemma 2.1.7 gives the desired result. $\square$

It should be noted that Kirchberg's Theorem for $C^*(\mathbb{F}_n)$ follows from Theorem 2.1.9. To show this fact, we will need the notion of a retract for operator systems. We will say that an operator system $\mathcal{S}$ is a **retract** of an operator system $\mathcal{T}$ if there are ucp maps $\psi : \mathcal{S} \to \mathcal{T}$ and $\chi : \mathcal{T} \to \mathcal{S}$ such that $\chi \circ \psi = \mathrm{id}_{\mathcal{S}}$.

Lemma 2.1.10. *Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ be operator systems, and let $\tau_1, \tau_2 \in \{\min, c, \max\}$. For $i = 1, 2$, suppose that $\mathcal{S}_i$ is a retract of $\mathcal{T}_i$. If $\mathcal{T}_1 \otimes_{\tau_1} \mathcal{T}_2 = \mathcal{T}_1 \otimes_{\tau_2} \mathcal{T}_2$ completely order isomorphically (respectively, order isomorphically), then $\mathcal{S}_1 \otimes_{\tau_1} \mathcal{S}_2 = \mathcal{S}_1 \otimes_{\tau_2} \mathcal{S}_2$ completely order isomorphically (respectively, order isomorphically).*

Proof. Since $\min \leq c \leq \max$ as operator system tensor products, we may assume that $\tau_1 \leq \tau_2$. Since $\mathcal{S}_i$ is a retract of $\mathcal{T}_i$, there are ucp maps $\varphi_i : \mathcal{S}_i \to \mathcal{T}_i$ and $\psi_i : \mathcal{T}_i \to \mathcal{S}_i$ such that $\psi_i \circ \varphi_i = \mathrm{id}_{\mathcal{S}_i}$. For each $j = 1, 2$, by functoriality of τ_j, the maps $\varphi_1 \otimes \varphi_2 : \mathcal{S}_1 \otimes_{\tau_j} \mathcal{S}_2 \to \mathcal{T}_1 \otimes_{\tau_j} \mathcal{T}_2$ and $\psi_1 \otimes \psi_2 : \mathcal{T}_1 \otimes_{\tau_j} \mathcal{T}_2 \to \mathcal{S}_1 \otimes_{\tau_j} \mathcal{S}_2$ are ucp. Moreover, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{T}_1 \otimes_{\tau_1} \mathcal{T}_2 \arrow[r, "\mathrm{id}_{\mathcal{T}_1} \otimes \mathrm{id}_{\mathcal{T}_2}"] & \mathcal{T}_1 \otimes_{\tau_2} \mathcal{T}_2 \arrow[d, "\psi_1 \otimes \psi_2"] \\
\mathcal{S}_1 \otimes_{\tau_1} \mathcal{S}_2 \arrow[u, "\varphi_1 \otimes \varphi_2"] \arrow[r, "\mathrm{id}_{\mathcal{S}_1} \otimes \mathrm{id}_{\mathcal{S}_2}"] & \mathcal{S}_1 \otimes_{\tau_2} \mathcal{S}_2
\end{tikzcd}$$

By assumption, the map $\mathrm{id} : \mathcal{T}_1 \otimes_{\tau_1} \mathcal{T}_2 \to \mathcal{T}_1 \otimes_{\tau_2} \mathcal{T}_2$ is completely positive (respectively, positive). Thus, $\mathrm{id} : \mathcal{S}_1 \otimes_{\tau_1} \mathcal{S}_2 \to \mathcal{S}_1 \otimes_{\tau_2} \mathcal{S}_2$ is completely positive (respectively, positive). The result follows. $\square$

For the next lemma, we define the operator system $\mathcal{S}_n \subseteq C^*(\mathbb{F}_n)$ to be

$$\mathcal{S}_n = \mathrm{span}\,\{1, w_1, ..., w_n, w_1^*, ..., w_n^*\},$$

where $w_1, ..., w_n$ are the generators of F_n.

Lemma 2.1.11. *Let $n \geq 2$.*

1. *$C^*(\mathbb{F}_n)$ is a retract of $\mathcal{U}_{nc}(n)$.*
2. *$\mathcal{S}_n$ is a retract of $\mathcal{V}_n$.*

Proof. To prove (1), we note that the diagonal matrix

$$\begin{pmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{pmatrix} \in M_n(C^*(\mathbb{F}_n))$$

is unitary. Hence, there is a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \to C^*(\mathbb{F}_n)$ such that $\pi(u_{ij}) = 0$ for $i \neq j$ and $\pi(u_{ii}) = w_i$. Then $\pi \otimes \mathrm{id}_{\mathcal{S}} : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S} \to C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}$ is ucp, while $\mathrm{id} : \mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} \to \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$ is ucp by Proposition 2.1.5 and Lemma 2.1.7. We let $U_1 = U := (u_{ij}) \in M_n(\mathcal{U}_{nc}(n))$. For $2 \leq i \leq n$, let U_i be the conjugation of U by a permutation matrix such that the $(1,1)$-entry of U_i is u_{ii}. Then each $U_i \in M_n(\mathcal{U}_{nc}(n))$ is unitary, so by the universal property for $C^*(\mathbb{F}_n)$, there is a unital $*$-homomorphism $\rho : C^*(\mathbb{F}_n) \to M_n(\mathcal{U}_{nc}(n))$ such that $\rho(w_i) = U_i$ for all $1 \leq i \leq n$. Compressing to the $(1,1)$-entry in $M_n(\mathcal{U}_{nc}(n))$ gives rise to a ucp map $\psi : C^*(\mathbb{F}_n) \to \mathcal{U}_{nc}(n)$ such that $\psi(w_i) = u_{ii}$ for all $1 \leq i \leq n$. Since $\pi \circ \psi(w_i) = w_i$ and since w_i is unitary, it follows that w_i lies in the multiplicative domain of $\pi \circ \psi$. Since $C^*(\mathbb{F}_n)$ is generated by $\{w_1, ..., w_n\}$, it follows that $\pi \circ \psi$ is multiplicative on $C^*(\mathbb{F}_n)$. The fact that $\pi \circ \psi(w_i) = w_i$ for all i forces $\pi \circ \psi = \mathrm{id}_{C^*(\mathbb{F}_n)}$. Thus, (1) holds.

For (2), since $\psi(w_i) = u_{ii} \in \mathcal{V}_n$ for all $1 \leq i \leq n$, we have $\psi(\mathcal{S}_n) \subseteq \mathcal{V}_n$. Clearly $\pi(\mathcal{V}_n) = \mathcal{S}_n$. Since $\pi \circ \psi = \mathrm{id}_{C^*(\mathbb{F}_n)}$, it follows that $\mathcal{S}_n$ is a retract of $\mathcal{V}_n$ via the maps $\psi_{|\mathcal{S}_n} : \mathcal{S}_n \to \mathcal{V}_n$ and $\pi_{|\mathcal{V}_n} : \mathcal{V}_n \to \mathcal{S}_n$. $\square$

Theorem 2.1.12. *Let $n \geq 2$, and let $\mathcal{S}$ be an operator system with property $\mathfrak{V}_n$. Then $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{S} = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}$.*

Proof. By Lemma 2.1.11, $C^*(\mathbb{F}_n)$ is a retract of $\mathcal{U}_{nc}(n)$. Applying Lemma 2.1.10, since $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{S} = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{S}$, it follows that $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{S} = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{S}$, which completes the proof. □

Corollary 2.1.13. (Kirchberg's Theorem, [42]) *Let $n \geq 2$. Then $C^*(\mathbb{F}_n)$ has the LLP. In other words, $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{B}(\mathcal{H}) = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{B}(\mathcal{H})$.*

Using Theorem 2.1.12, it is possible to characterize unital C^*-algebras having the WEP and operator systems having the DCEP in terms of tensor products with $\mathcal{V}_2$.

Theorem 2.1.14. *Let $\mathcal{A}$ be a unital C^*-algebra. The following are equivalent.*

1. *$\mathcal{A}$ has the WEP.*
2. *$\mathcal{A}$ has property $\mathfrak{V}$.*
3. *$\mathcal{A}$ has property $\mathfrak{V}_n$ for some $n \geq 2$.*
4. *$\mathcal{A} \otimes_{\min} \mathcal{V}_n = \mathcal{A} \otimes_{\max} \mathcal{V}_n$ for all $n \geq 2$.*
5. *$\mathcal{A} \otimes_{\min} \mathcal{V}_n = \mathcal{A} \otimes_{\max} \mathcal{V}_n$ for some $n \geq 2$.*

Proof. Clearly (2) implies (3) and (4) implies (5), while (2) implies (4) by Proposition 2.1.5. Similarly, (3) implies (5). Suppose that $\mathcal{A}$ has the WEP. By Theorem 1.3.5, $\mathcal{A}$ is $(\mathrm{el}, \max)$-nuclear. By Theorem 1.3.4, each $\mathcal{V}_n$ having the OSLLP implies that each $\mathcal{V}_n$ is $(\min, \mathrm{er})$-nuclear. Hence,

$$\mathcal{V}_n \otimes_{\min} \mathcal{A} = \mathcal{V}_n \otimes_{\mathrm{er}} \mathcal{A} = \mathcal{A} \otimes_{\mathrm{el}} \mathcal{V}_n = \mathcal{A} \otimes_{\max} \mathcal{V}_n = \mathcal{V}_n \otimes_{\max} \mathcal{A}.$$

By Proposition 2.1.5 and the fact that $n \geq 2$ was arbitrary, we conclude that $\mathcal{A}$ has property $\mathfrak{V}$. This shows that (1) implies (2).

Finally, we prove that (5) implies (1). Suppose that $\mathcal{A} \otimes_{\min} \mathcal{V}_n = \mathcal{A} \otimes_{\max} \mathcal{V}_n$ for some $n \geq 2$. Then by Lemma 2.1.7, we have $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{A} = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{A}$. Using Theorem 2.1.12, we have $C^*(\mathbb{F}_n) \otimes_{\min} \mathcal{A} = C^*(\mathbb{F}_n) \otimes_{\max} \mathcal{A}$. As $\mathbb{F}_\infty$ embeds as a subgroup into $\mathbb{F}_n$, by Proposition 1.4.1, it follows that there are ucp maps $\Phi : C^*(\mathbb{F}_\infty) \to C^*(\mathbb{F}_n)$ and $\Psi : C^*(\mathbb{F}_n) \to C^*(\mathbb{F}_\infty)$ with $\Psi \circ \Phi = \mathrm{id}$. By Lemma 2.1.10, we have $C^*(\mathbb{F}_\infty) \otimes_{\min} \mathcal{A} = C^*(\mathbb{F}_\infty) \otimes_{\max} \mathcal{A}$. By Theorem 1.3.6, $\mathcal{A}$ has the WEP. □

There is a similar characterization for operator systems with the DCEP.

Theorem 2.1.15. *Let $\mathcal{S}$ be an operator system. The following are equivalent.*

1. *$\mathcal{S}$ has the DCEP.*
2. *$\mathcal{S} \otimes_{\min} \mathcal{V}_n = \mathcal{S} \otimes_c \mathcal{V}_n$ for all $n \geq 2$.*
3. *$\mathcal{S} \otimes_{\min} \mathcal{V}_n = \mathcal{S} \otimes_c \mathcal{V}_n$ for some $n \geq 2$.*

Proof. Assume that $\mathcal{S}$ has the DCEP. By Theorem 1.3.6, $\mathcal{S}$ is (el, c)-nuclear, while $\mathcal{V}_n$ is $(\min, \mathrm{er})$-nuclear. It follows that $\mathcal{S} \otimes_{\min} \mathcal{V}_n = \mathcal{S} \otimes_c \mathcal{V}_n$ for all $n \geq 2$. Hence, (1) implies (2). Clearly (2) implies (3). If (3) is true, then by Lemma 2.1.7 and by Theorem 2.1.12, we must have $\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_n) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_n)$. Since $C^*(\mathbb{F}_\infty)$ is a retract of $C^*(\mathbb{F}_n)$, using Lemma 2.1.10 gives $\mathcal{S} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{S} \otimes_{\max} C^*(\mathbb{F}_\infty)$. Applying Theorem 1.3.6 shows that $\mathcal{S}$ has the DCEP, so that (1) is true. □

2.2 Relating the Brown operator system $\mathcal{V}_n$ to Kirchberg's conjecture

The proof of Theorem 2.1.12 shows that $C^*(\mathbb{F}_n)$ is a retract of $\mathcal{U}_{nc}(n)$ via ucp maps. Using this fact allows for a connection between $\mathcal{U}_{nc}(n)$ and Kirchberg's conjecture.

Theorem 2.2.1. *If $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n) = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ for some $n \geq 2$, then Kirchberg's conjecture is valid.*

Proof. It is well known that Kirchberg's conjecture is true if and only if it holds for some $n \in \mathbb{N}$ with $n \geq 2$. Now, if $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n) = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$, then combining Lemmas 2.1.10 and 2.1.11 yields the complete order isomorphism $C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n)$. □

The link between Kirchberg's conjecture and the WEP allows us to prove the converse of Theorem 2.2.1. In other words, while the assumption that $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n) = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ for some $n \geq 2$ appears to be slightly stronger than Kirchberg's conjecture, it is in fact equivalent to Kirchberg's conjecture.

Theorem 2.2.2. *The following statements are equivalent.*

1. $\mathcal{V}_2 \otimes_{\min} \mathcal{V}_2 = \mathcal{V}_2 \otimes_c \mathcal{V}_2$.

2. $\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2) = \mathcal{U}_{nc}(2) \otimes_{\max} \mathcal{U}_{nc}(2)$.

3. $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_2)$.

4. $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$.

5. *Connes' embedding problem has a positive answer.*

Proof. The equivalence of (4) and (5) is part of Theorem 1.8.5. Note that if (3) holds, then since $C^*(\mathbb{F}_\infty)$ is a retract of $C^*(\mathbb{F}_2)$, (4) also holds by the same argument as in the proof of Theorem 2.1.14. Clearly $\mathbb{F}_2$ embeds into $\mathbb{F}_\infty$ so that, by Proposition 1.4.1, $C^*(\mathbb{F}_2)$ is a retract of $C^*(\mathbb{F}_\infty)$. Hence, (4) implies (3). Using Lemma 2.1.7 shows that (1) implies (2), while Theorem 2.2.1 shows that (2) implies (3). Assuming (4) is true, it follows that $C^*(\mathbb{F}_\infty)$ has the WEP by Theorem 1.3.6. Then Theorem 1.3.7 shows that any operator system $\mathcal{S}$ that is $(\min, \mathrm{er})$-nuclear satisfies $\mathcal{S} \otimes_{\min} \mathcal{S} = \mathcal{S} \otimes_c \mathcal{S}$. By Lemma 2.1.8 and Theorem 1.3.4, $\mathcal{V}_2$ is $(\min, \mathrm{er})$-nuclear. Therefore, $\mathcal{V}_2 \otimes_{\min} \mathcal{V}_2 = \mathcal{V}_2 \otimes_c \mathcal{V}_2$, as required. □

Because $\mathcal{S}_n$ is a retract of $\mathcal{V}_n$, we can prove the following.

Proposition 2.2.3. *For all $n, m \geq 2$, $\mathcal{V}_n \otimes_c \mathcal{V}_m \neq \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$.*

Proof. By Theorem 1.3.8, we have $\mathcal{S}_n \otimes_c \mathcal{S}_m \neq \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$ for all $n, m \geq 2$. Hence, if $\mathcal{V}_n \otimes_c \mathcal{V}_m = \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$, then by Lemmas 2.1.10 and 2.1.11, we have $\mathcal{S}_n \otimes_c \mathcal{S}_m = \mathcal{S}_n \otimes_{\max} \mathcal{S}_m$, which is a contradiction. □

Corollary 2.2.4. *For all $n, m \geq 2$, $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m) \neq C_e^*(\mathcal{V}_n) \otimes_{\max} C_e^*(\mathcal{V}_m)$.*

Proof. Suppose that $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m) = C_e^*(\mathcal{V}_n) \otimes_{\max} C_e^*(\mathcal{V}_m)$. The latter C^*-algebra is $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m)$. Applying Proposition 2.1.6, $\mathcal{V}_n \otimes_c \mathcal{V}_m$ is completely order isomorphic to its inclusion in $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m)$. The operator system $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$ is evidently completely order isomorphic to its inclusion in $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$. Thus, $\mathcal{V}_n \otimes_c \mathcal{V}_m = \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$, contradicting Proposition 2.2.3. □

Corollary 2.2.5. *Let $n, m \geq 2$. Let $\{u_{ij}\}_{i,j=1}^n$ be the set of generators of the copy of $\mathcal{V}_n$ in $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$, and let $\{v_{k\ell}\}_{k,\ell=1}^m$ be the set of generators of the copy of $\mathcal{V}_m$ in $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. Define $U, V \in M_{nm}(C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m))$ by $U = (u_{ij})_{i,j=1}^n \otimes 1_m$ and $V = 1_n \otimes (v_{k\ell})_{k,\ell=1}^m$. Then neither U nor V are unitary.*

Proof. Suppose that one of U or V is unitary. We will adapt the proof of [22, Lemma 2.4] to show that U and V must commute. Let $U_0 = (u_{ij}) \in M_n(\mathcal{V}_n)$ and $V_0 = (v_{k\ell}) \in M_m(\mathcal{V}_m)$. Since these matrices are contractions, we may define

$$P = \begin{pmatrix} 1_n & U_0 & U_0 \\ U_0^* & 1_n & 1_n \\ U_0^* & 1_n & 1_n \end{pmatrix} \in M_3(M_n(\mathcal{V}_n))_+$$

and

$$Q = \begin{pmatrix} 1_m & 1_m & V_0 \\ 1_m & 1_m & V_0 \\ V_0^* & V_0^* & 1_m \end{pmatrix} \in M_3(M_m(\mathcal{V}_m))_+.$$

It follows that $P \otimes Q \in M_9(M_n(\mathcal{V}_n) \otimes_{\max} M_m(\mathcal{V}_m))_+$. Let $A : \mathbb{C}^3 \to \mathbb{C}^9$ be defined by $Ae_1 = f_1$, $Ae_2 = f_4$ and $Ae_3 = f_9$, where $\{e_1, e_2, e_3\}$ is the standard orthonormal basis for $\mathbb{C}^3$ and $\{f_1, ..., f_9\}$ is the standard orthonormal basis for $\mathbb{C}^9$. Then the matrix $W = A^*(P \otimes Q)A$ belongs to $M_3(M_n(\mathcal{V}_n) \otimes_{\max} M_m(\mathcal{V}_m))_+$. Moreover, we have

$$W = \begin{pmatrix} 1_n \otimes 1_m & U_0 \otimes 1_m & U_0 \otimes V_0 \\ U_0^* \otimes 1_m & 1_n \otimes 1_m & 1_n \otimes V_0 \\ U_0^* \otimes V_0^* & 1_n \otimes V_0^* & 1_n \otimes 1_m \end{pmatrix}.$$

We may write

$$W = \begin{pmatrix} 1_n \otimes 1_m & X \\ X^* & C \end{pmatrix},$$

where $X = \begin{pmatrix} U_0 \otimes 1_m & U_0 \otimes V_0 \end{pmatrix}$ and $C = \begin{pmatrix} 1_n \otimes 1_m & 1_n \otimes V_0 \\ 1_n \otimes V_0^* & 1_n \otimes 1_m \end{pmatrix}$. Similarly, we may write

$$W = \begin{pmatrix} D & Y \\ Y^* & 1_n \otimes 1_m \end{pmatrix},$$

where $D = \begin{pmatrix} 1_n \otimes 1_m & U_0 \otimes 1_m \\ U_0^* \otimes 1_m & 1_n \otimes 1_m \end{pmatrix}$ and $Y = \begin{pmatrix} U_0 \otimes V_0 \\ 1_n \otimes V_0 \end{pmatrix}$. Since W is positive, the matrices $C - X^*X$ and $D - YY^*$ must be positive in $M_2(M_n(\mathcal{V}_n) \otimes_{\max} M_m(\mathcal{V}_m))$. We can write

$$C - X^*X = \begin{pmatrix} 1_n \otimes 1_m - (U_0^* \otimes 1_m)(U_0 \otimes 1_m) & 1_n \otimes V_0 - (U_0^* \otimes 1_m)(U_0 \otimes V_0) \\ * & * \end{pmatrix}$$

and

$$D - YY^* = \begin{pmatrix} * & U_0 \otimes 1_m - (U_0 \otimes V_0)(1_n \otimes V_0^*) \\ * & 1_n \otimes 1_m - (1_n \otimes V_0)(1_n \otimes V_0^*) \end{pmatrix}.$$

So, if $U_0 \otimes 1_m$ is unitary in $M_{nm}(C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m))$, then the $(1,2)$ entry of $C - X^*X$ is 0. Multiplying this entry by $U_0 \otimes 1_m$ yields $(U_0 \otimes 1_m)(1_n \otimes V_0) = U_0 \otimes V_0$, where the product is being done in $M_{nm}(C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m))$. Similarly, if $1_n \otimes V_0$ is unitary in $M_{nm}(C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_n))$, then the $(1,2)$ entry of $D - YY^*$ is 0, and multiplying by $1_n \otimes V_0$ yields $(U_0 \otimes 1_m)(1_n \otimes V_0) = U_0 \otimes V_0$.

To show that $(1_n \otimes V_0)(U_0 \otimes 1_m) = U_0 \otimes V_0$, we replace the matrices P and Q with $R = \begin{pmatrix} 1_n & 1_n & U_0 \\ 1_n & 1_n & U_0 \\ U_0^* & U_0^* & 1_n \end{pmatrix}$ and $S = \begin{pmatrix} 1_m & V_0 & V_0 \\ V_0^* & 1_m & 1_m \\ V_0^* & 1_m & 1_m \end{pmatrix}$, respectively. Then

$$Z := A^*(R \otimes S)A = \begin{pmatrix} 1_n \otimes 1_m & 1_n \otimes V_0 & U_0 \otimes V_0 \\ 1_n \otimes V_0^* & 1_n \otimes 1_m & U_0 \otimes 1_m \\ U_0^* \otimes V_0^* & U_0^* \otimes 1_m & 1_n \otimes 1_m \end{pmatrix} \in M_3(M_n(\mathcal{V}_n) \otimes_{\max} M_m(\mathcal{V}_m))_+.$$

A similar argument to the above shows that, if either of U or V is unitary, then we must have $(1_n \otimes V_0)(U_0 \otimes 1_m) = U_0 \otimes V_0$. Hence, if either of U or V are unitary, then U and V must commute. But if U is unitary and $UV = VU$, then multiplying on both sides by U^* gives $VU^* = U^*V$, so that U and V $*$-commute. Similarly, if V is unitary, then U and V must $*$-commute.

We claim that, if one of U or V is unitary, then the set $\{u_{ij} \otimes 1_{\mathcal{V}_m}\}_{i,j=1}^n$ will $*$-commute with the set $\{1_{\mathcal{V}_n} \otimes v_{k\ell}\}_{k,\ell=1}^m$ in $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$. To show this, we adapt the proof of [11, Proposition 3.1]. Assume that $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ is faithfully represented on a Hilbert space $\mathcal{H}$, and let $\widehat{u}_{ij}$ be the image of $u_{ij} \otimes 1_{\mathcal{V}_m}$ and $\widehat{v}_{k\ell}$ be the image of $1_{\mathcal{V}_n} \otimes v_{k\ell}$ under this representation. We may identify the image $\widehat{U_0}$ of U_0 as a unitary on $\mathbb{C}^n \otimes \mathcal{H}$ and the image $\widehat{V_0}$ of V_0 as a unitary on $\mathcal{H} \otimes \mathbb{C}^m$. Then the image of U is $\widehat{U_0} \otimes I_m$ and the image of V is $I_n \otimes \widehat{V_0}$. On a vector of the form $e_j \otimes h \otimes e_k$, one has

$$(\widehat{U_0} \otimes I_m)(I_n \otimes \widehat{V_0})(e_j \otimes h \otimes e_\ell) = \sum_{i,j,k,\ell} e_i \otimes \widehat{u}_{ij}\widehat{v}_{k\ell} \otimes e_k$$

and

$$(I_n \otimes \widehat{V_0})(\widehat{U_0} \otimes I_m)(e_j \otimes h \otimes e_\ell) = \sum_{i,j,k,\ell} e_i \otimes \widehat{v}_{k\ell}\widehat{u}_{ij} e_k.$$

Thus, we must have that $\widehat{u}_{ij}$ commutes with $\widehat{v}_{k\ell}$, so that $u_{ij} \otimes 1_{\mathcal{V}_m}$ commutes with $1_{\mathcal{V}_n} \otimes v_{k\ell}$ in $C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$. Since $\widehat{U_0} \otimes I_m$ and $I_n \otimes \widehat{V_0}$ also $*$-commute, a similar argument shows that $u_{ij} \otimes 1_{\mathcal{V}_m}$ $*$-commutes with $1_{\mathcal{V}_n} \otimes v_{k\ell}$. Therefore, the identity maps $\mathrm{id} : \mathcal{V}_n \to \mathcal{V}_n \subseteq C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ and $\mathrm{id} : \mathcal{V}_m \to \mathcal{V}_m \subseteq C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ are ucp maps with commuting ranges. Thus, the identity map $\mathrm{id} : \mathcal{V}_n \otimes_c \mathcal{V}_m \to \mathcal{V}_n \otimes_{\max} \mathcal{V}_m \subseteq C_e^*(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ is ucp, contradicting Proposition 2.2.3. $\square$

2.3 Order isomorphisms of tensor products of $\mathcal{V}_n$

Using the probabilistic correlation sets as a guide, we will consider a new collection of correlation sets that correspond to the Brown operator system $\mathcal{V}_n$. In order to link these so-called unitary correlations back to Connes' embedding problem, we will relate the unitary correlations to the probabilistic correlation sets $C_t(n, 2)$ for $t \in \{q, qs, qa, qc\}$.

The special case of two outputs for probabilistic correlations involves the C^*-algebra $C^*(*_n\mathbb{Z}_2)$. Following the notation in [22], we let h_i be the generator of the i-th copy of $\mathbb{Z}_2$ inside of $C^*(*_n\mathbb{Z}_2)$. Each h_i is a self-adjoint unitary. We let $NC(n)$ be the operator system generated by $\{h_1, ..., h_n\}$ inside of $C^*(*_n\mathbb{Z}_2)$.

Proposition 2.3.1. (Farenick-Kavruk-Paulsen-Todorov, [22]) *If $X_1, ..., X_n \in \mathcal{B}(\mathcal{H})$ are hermitian contractions, then there is a unique ucp map $\gamma : NC(n) \to \mathcal{B}(\mathcal{H})$ given by $\gamma(h_i) = X_i$ for all $1 \leq i \leq n$.*

The isomorphism $C^*(*_n\mathbb{Z}_2) \simeq \mathcal{F}_{n,2} := *_n\ell^2_\infty$ is implemented by the mapping

$$h_i \mapsto p_i - q_i,$$

where p_i is the element $(1, 0)$ in the i-th copy of ℓ^∞_2, and q_i is the element $(0, 1)$ in the i-th copy of ℓ^∞_2. In this way, $NC(n)$ is completely order isomorphic to the operator system $\mathcal{S}_{n,2}$ from Section 1.7. A few more facts about $NC(n)$ are required for our purposes.

Proposition 2.3.2. (Farenick-Kavruk-Paulsen-Todorov, [22, Proposition 5.7]) *$NC(n)$ is a retract of $\mathcal{S}_n$, where $\mathcal{S}_n = span\ \{1, w_1, ..., w_n, w_1^*, ..., w_n^*\} \subseteq C^*(\mathbb{F}_n)$.*

Using the complete order isomorphism between $NC(n)$ and the operator system $\mathcal{S}_{n,2}$, along with Theorems 1.7.3 and 1.7.10, we easily obtain the following.

Proposition 2.3.3. *For $n \geq 2$, $C_{qa}(n, 2) = C_{qc}(n, 2)$ if and only if the identity map $id : NC(n) \otimes_{\min} NC(n) \to NC(n) \otimes_c NC(n)$ is an order isomorphism.*

The first thing that relates $\mathcal{V}_n$ to $NC(n)$ is the following property:

Proposition 2.3.4. *For any $n \geq 2$, $NC(n)$ is a retract of $\mathcal{V}_n$.*

Proof. By Proposition 2.3.2, there are ucp maps $\eta : NC(n) \to \mathcal{S}_n$ and $\theta : \mathcal{S}_n \to NC(n)$ such that $\theta \circ \eta = \mathrm{id}_{NC(n)}$. By Lemma 2.1.11, there are ucp maps $\psi : \mathcal{S}_n \to \mathcal{V}_n$ and $\pi : \mathcal{V}_n \to \mathcal{S}_n$ with $\pi \circ \psi = \mathrm{id}_{\mathcal{S}_n}$. Then $\psi \circ \eta : NC(n) \to \mathcal{V}_n$ and $\theta \circ \pi : \mathcal{V}_n \to NC(n)$ are ucp maps satisfying $(\theta \circ \pi) \circ (\psi \circ \eta) = \mathrm{id}_{NC(n)}$. We conclude that $NC(n)$ is a retract of $\mathcal{V}_n$. $\square$

We wish to define correlation matrices with respect to $\mathcal{U}_{nc}(n)$ that are similar in nature to Tsirelson's correlation sets. A key component in linking the usual quantum correlation matrices with Kirchberg's conjecture is the fact that $C^*(\mathbb{F}_n)$ is RFD for every n. Here, we show that $\mathcal{U}_{nc}(n)$ also enjoys this property.

Theorem 2.3.5. *For any $n \geq 2$, $\mathcal{U}_{nc}(n)$ is RFD.*

Proof. The proof mimics the proof that $C^*(\mathbb{F}_n)$ is RFD (see [9, Theorem 7]). It is not hard to see that $\mathcal{U}_{nc}(n)$ is a separable C^*-algebra, so we may assume that $\mathcal{U}_{nc}(n) \subseteq \mathcal{B}(\mathcal{H})$ is faithfully represented on a separable infinite-dimensional Hilbert space $\mathcal{H}$. Hence, there are operators $U_{ij} \in \mathcal{B}(\mathcal{H})$ for $1 \leq i, j \leq n$ such that $\mathcal{U}_{nc}(n) \simeq C^*(\{U_{ij}\}_{i,j})$ via the mapping $u_{ij} \mapsto U_{ij}$. Let $(P_m)_{m=1}^\infty$ be a sequence of increasing projections with $\text{rank}(P_m) = m$ and $SOT\text{-}\lim_{m\to\infty} P_m = I$. Define $V_{m,ij} = P_m U_{ij} P_m$ and let $V_m = (V_{m,ij})$. Since $\text{rank}(P_m) = m$, we may identify $V_{m,ij} \in M_m$ for each i, j and hence $V_m \in M_n(M_m)$. Observe that

$$V_m = \begin{pmatrix} P_m & & \\ & \ddots & \\ & & P_m \end{pmatrix} U \begin{pmatrix} P_m & & \\ & \ddots & \\ & & P_m \end{pmatrix},$$

where $U = (U_{ij})$. Therefore, each V_m is a contraction. By Proposition 2.1.1, there exist unital $*$-homomorphisms $\pi_m : \mathcal{U}_{nc}(n) \to M_2(M_m)$ for each $m \in \mathbb{N}$ such that

$$X_{m,ij} := \pi_m(u_{ij}) = \begin{pmatrix} V_{m,ij} & (\sqrt{I - V_m V_m^*})_{ij} \\ (\sqrt{I - V_m^* V_m})_{ij} & -V_{m,ji}^* \end{pmatrix}$$

for all i, j. Since $V_{m,ij}^* = P_m U_{ij}^* P_m$, $SOT\text{-}\lim_{m\to\infty} V_m = U$ and $SOT\text{-}\lim_{m\to\infty} V_m^* = U^*$. Hence, every entry of V_m converges in SOT, so that

$$SOT\text{-}\lim_{m\to\infty} X_m = \begin{pmatrix} U_{ij} & 0 \\ 0 & -U_{ji}^* \end{pmatrix}.$$

Let F be any finite linear combination of words in the generators of $\mathcal{U}_{nc}(n)$. We similarly obtain

$$SOT\text{-}\lim_{m\to\infty} \pi_m(F) = \begin{pmatrix} F & 0 \\ 0 & F(\{-U_{ji}^*, -U_{ji}\}) \end{pmatrix},$$

where $F(\{-U_{ji}^*, -U_{ji}\})$ is the word obtained by replacing every occurrence of U_{ij} with $-U_{ji}^*$, and every occurrence of U_{ij}^* with $-U_{ji}$. Assume that F is norm 1. Then given $\varepsilon > 0$, there is $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, we have $\|F(\{X_{m,ij}, X_{m,ij}^*\})\| \geq 1 - \varepsilon$. Hence, $\pi := \bigoplus_{m\in\mathbb{N}} \pi_m$ is isometric on the dense subspace of linear combinations of words in the generators of $\mathcal{U}_{nc}(n)$. Since π must be continuous, π is isometric on $\mathcal{U}_{nc}(n)$. This shows that π is faithful and $\mathcal{U}_{nc}(n)$ is RFD. $\square$

Remark 2.3.6. *It is not hard to see that whenever $\mathcal{A}$ and $\mathcal{B}$ are RFD C^*-algebras, then $\mathcal{A} \otimes_{\min} \mathcal{B}$ is also RFD. Hence, $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(k)$ is RFD for every $n, k \geq 2$.*

As with $C^*(\mathbb{F}_n)$, we can reformulate Kirchberg's conjecture in terms of whether or not $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ is RFD. The proof is identical to the $C^*(\mathbb{F}_n)$ case [8, Proposition 7.4.4], and is omitted.

Theorem 2.3.7. *The following statements are equivalent.*

1. *(Kirchberg's Conjecture) $C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) = C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n)$ for all/some $n \geq 2$.*
2. *$\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n) = \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ for all/some $n \geq 2$.*
3. *$\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ is RFD for all/some $n \geq 2$.*

We will show below that (3) holds if we weaken the assumption of residual finite-dimensionality to being quasidiagonal. Recall that a unital C^*-algebra $\mathcal{A}$ is quasidiagonal if there is a net of ucp maps $\varphi_\lambda : \mathcal{A} \to M_{k(\lambda)}$ such that $\lim_\lambda \|\varphi_\lambda(a)\| = \|a\|$ and $\lim_\lambda \|\varphi_\lambda(ab) - \varphi_\lambda(a)\varphi_\lambda(b)\| = 0$ for all $a, b \in \mathcal{A}$. It is easy to see that every RFD C^*-algebra is QD.

Theorem 2.3.8. *For every $n \geq 2$, $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ is QD.*

Proof. The proof is similar to the proof for $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n)$ (see [8, Proposition 7.4.5]). Let $\pi : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n) \to \mathcal{B}(\mathcal{H})$ be a faithful representation on a Hilbert space $\mathcal{H}$. Let $U = (U_{ij})$ be the matrix of generators of $\mathcal{U}_{nc}(n) \otimes 1$, and let $V = (V_{ij})$ be the matrix of generators of $1 \otimes \mathcal{U}_{nc}(n)$, so that each $U_{ij}, V_{ij} \in \mathcal{B}(\mathcal{H})$. The nature of the max tensor product forces the U_{ij}'s and $V_{k\ell}$'s to $*$-commute. The unitary group of $\mathcal{B}(\mathcal{H}^{(n)})$ is path connected by the Borel functional calculus. Hence, there are norm-continuous functions $u, v : [0,1] \to \mathcal{B}(\mathcal{H}^{(n)})$ such that $u(0) = \mathcal{I}_{\mathcal{H}^{(n)}} = v(0)$, $u(1) = U$ and $v(1) = V$. Using the Borel functional calculus, we can arrange to have $u(t) \in W^*(U)$ and $v(t) \in W^*(V)$ for all $t \in [0,1]$. The entries of U and V $*$-commute, so this must also hold for the entries of $p(U, U^*)$ and $q(V, V^*)$ for any $*$-polynomials p, q. Taking limits, one sees that the entries of $u(t) \in \mathcal{B}(\mathcal{H}^{(n)})$ must $*$-commute with the entries of $v(t) \in \mathcal{B}(\mathcal{H}^{(n)})$ for all $t \in [0,1]$. Since $u(t)$ and $v(t)$ are unitary with $*$-commuting entries, there is a unique $*$-homomorphism $\pi_t : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n) \to \mathcal{B}(\mathcal{H})$ with $\pi_t(u_{ij} \otimes 1) = (u(t))_{ij}$ and $\pi_t(1 \otimes v_{ij}) = (v(t))_{ij}$. As π_0 is the trivial representation onto $\mathbb{C}I_{\mathcal{H}}$ and $\pi_1 = \pi$, we see that π is homotopic to the trivial representation. Since π is injective and $\pi_0(\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)) = \mathbb{C}$ is obviously QD, by Theorem 1.3.11, $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ is QD. $\square$

We are now in a position to define our unitary correlation sets. As in the usual setting, we will consider a *tensor product* model as well as a *commuting* model. For $n \geq 2$ and a unitary $U = (U_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$, we let $\mathfrak{B}_n(U) = \{I_{\mathcal{H}}\} \cup \{U_{ij}, U_{ij}^*\}_{i,j=1}^n$. We define $UC_q(n_1, n_2)$ to be the set of all $(2n_1^2+1)(2n_2^2+1)$-tuples of the form

$$(\langle (X \otimes Y)\xi, \xi\rangle)_{X \in \mathfrak{B}_{n_1}(U),\, Y \in \mathfrak{B}_{n_2}(V)},$$

where $U \in M_{n_1}(\mathcal{B}(\mathcal{H}_A))$ and $V \in M_{n_2}(\mathcal{B}(\mathcal{H}_B))$, $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert spaces, and $\xi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a unit vector. We define the possibly larger set $UC_{qs}(n_1, n_2)$ to be the set of all correlations of the same form as for $UC_q(n_1, n_2)$, except that we allow the Hilbert spaces to be infinite-dimensional. For convenience, we will also define $UC_{qa}(n_1, n_2)$ to be the closure of $UC_q(n_1, n_2)$. For the commuting unitary correlation sets, we define $UC_{qc}(n_1, n_2)$ to be the set of all $(2n_1^2+1)(2n_2^2+1)$-tuples of the form

$$(\langle XY\xi, \xi\rangle)_{X \in \mathfrak{B}_{n_1}(U),\, Y \in \mathfrak{B}_{n_2}(V)},$$

where $U \in M_{n_1}(\mathcal{B}(\mathcal{H}))$ and $V \in M_{n_2}(\mathcal{B}(\mathcal{H}))$ are unitaries, $\mathcal{H}$ is a Hilbert space, $\xi \in \mathcal{H}$ is a unit vector, and $XY = YX$ for all $X \in \mathfrak{B}_{n_1}(U)$ and $Y \in \mathfrak{B}_{n_2}(V)$. By definition of $\mathfrak{B}_{n_1}(U)$ and $\mathfrak{B}_{n_2}(V)$, it follows that the U_{ij}'s and $V_{k\ell}$'s $*$-commute. For convenience, we denote by $\mathcal{G}_{n_1,n_2}$ the set of generators of $\mathcal{V}_{n_1} \otimes \mathcal{V}_{n_2}$ of the form $x \otimes y$, where $x \in \{1\} \cup \{u_{ij}, u_{ij}^*\}_{i,j=1}^{n_1}$ and $y \in \{1\} \cup \{v_{k\ell}, v_{k\ell}^*\}_{k,\ell=1}^{n_2}$. By the correspondence between GNS representations and states,

$$UC_{qc}(n_1, n_2) = \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathcal{S}(\mathcal{U}_{nc}(n_1) \otimes_{\max} \mathcal{U}_{nc}(n_2))\}.$$

By Proposition 2.1.6, the inclusion $\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2} \subseteq \mathcal{U}_{nc}(n_1) \otimes_{\max} \mathcal{U}_{nc}(n_2)$ is a complete order embedding. Therefore, we may also write

$$UC_{qc}(n_1, n_2) = \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2})\}.$$

Similarly, we may define

$$UC_{qmin}(n_1, n_2) = \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2})\}.$$

Applying Corollary 1.7.5 and injectivity of the minimal tensor product, it is not hard to see that

$$UC_q(n_1, n_2) = \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathrm{Fin}(\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2))\}.$$

These unitary correlation sets have similar properties to the quantum correlation sets.

Proposition 2.3.9. *For every $n_1, n_2 \geq 2$,*

$$UC_q(n_1, n_2) \subseteq UC_{qs}(n_1, n_2) \subseteq UC_{qa}(n_1, n_2) \subseteq UC_{qc}(n_1, n_2),$$

and each of these sets is convex. Moreover, $UC_{qc}(n_1, n_2)$ is closed.

Proof. Since the state space of any operator system is convex, it is easy to see that each set above is convex. Clearly $UC_q(n_1, n_2) \subseteq UC_{qs}(n_1, n_2)$. Every element of $UC_{qs}(n_1, n_2)$ corresponds to a state on $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2}$, which extends to a state on $\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2)$ by the Hahn-Banach theorem. By Theorem 1.7.7, the set $\text{Fin}(\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2))$ is w^*-dense in $\mathcal{S}(\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2))$, so that each element of $UC_{qs}(n_1, n_2)$ is also in $UC_{qa}(n_1, n_2)$. To show that $UC_{qa}(n_1, n_2) \subseteq UC_{qc}(n_1, n_2)$, it suffices to show that $UC_{qc}(n_1, n_2)$ is closed. To that end, let $((s_p(x)_{x \in \mathcal{G}_{n_1,n_2}})_{p=1}^{\infty}$ be a sequence in $UC_{qc}(n_1, n_2)$ that converges, where $(s_p)_{p=1}^{\infty} \subseteq \mathcal{S}(\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2})$. The mapping $s : \mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2} \to \mathbb{C}$ given by $s(x) = \lim_{p \to \infty} s_p(x)$ for all $x \in \mathcal{G}_{n_1,n_2}$ extends to a linear functional. It follows that $s = w^*\text{-}\lim_{p \to \infty} s_p$. Since the state space on an operator system is w^*-closed, we see that $s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2})$ so that $UC_{qc}(n_1, n_2)$ is closed. $\square$

Before we link these unitary correlation sets to Connes' embedding problem, it will be helpful to have a better description of $UC_{qa}(n_1, n_2)$.

Lemma 2.3.10. *For each $n_1, n_2 \geq 2$,*

$$UC_{qa}(n_1, n_2) = \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2})\}.$$

In particular, $UC_{qa}(n_1, n_2) = UC_{qmin}(n_1, n_2)$.

Proof. Note that $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2}$ is completely order isomorphic to its inclusion in $\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2)$. Since $\mathcal{S}(\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2})$ is w^*-closed, the proof of Proposition 2.3.9 shows that

$$UC_{qa}(n_1, n_2) \subseteq \{(s(x))_{x \in \mathcal{G}_{n_1,n_2}} : s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2})\}.$$

Conversely, let $s \in \mathcal{S}(\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2})$. By the Hahn-Banach theorem we may extend s to a state on $\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2)$. Since $\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2)$ is RFD, by Theorem 1.7.7, s can be approximated pointwise by elements of $\text{Fin}(\mathcal{U}_{nc}(n_1) \otimes_{\min} \mathcal{U}_{nc}(n_2))$. Restricting to the set $\mathcal{G}_{n_1,n_2}$ yields a net of states whose images on the set $\mathcal{G}_{n_1,n_2}$ are elements of $UC_q(n_1, n_2)$ by Corollary 1.7.5. Since this net of states converges pointwise to s, we see that $(s(x))_{x \in \mathcal{G}_{n_1,n_2}} \in UC_{qa}(n_1, n_2)$, as required. $\square$

Using Lemma 2.3.10 allows us to formulate the problem of deciding whether the sets $UC_{qa}(n_1, n_2)$ and $UC_{qc}(n_1, n_2)$ are equal in terms of $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2}$ and $\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$.

Lemma 2.3.11. *Let $n_1, n_2 \geq 2$. Then $UC_{qa}(n_1, n_2) = UC_{qc}(n_1, n_2)$ if and only if $id : \mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2} \to \mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$ is an order isomorphism.*

Proof. If $UC_{qa}(n_1, n_2) = UC_{qc}(n_1, n_2)$, then by linearity the states on $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2}$ and $\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$ are the same. An element in an operator system is positive if and only if its image under each state is positive (see, for example, [49, Chapter 13]), so we conclude that $C_1^{\min}(\mathcal{V}_{n_1}, \mathcal{V}_{n_2}) = C_1^{\text{comm}}(\mathcal{V}_{n_1}, \mathcal{V}_{n_2})$. Therefore, $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2}$ and $\mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$ must be order isomorphic. Conversely, if $\text{id} : \mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2} \to \mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$ is an order isomorphism, then the positive elements are the same in the two operator systems, so the state spaces are identical. Restricting to the set $\mathcal{G}_{n_1,n_2}$, we obtain the equality $UC_{qa}(n_1, n_2) = UC_{qc}(n_1, n_2)$. $\square$

We are now ready for the main result of this section.

Theorem 2.3.12. *The following are equivalent.*

1. *Connes' embedding problem has a positive answer.*
2. *$UC_{qa}(n_1, n_2) = UC_{qc}(n_1, n_2)$ for all $n_1, n_2 \geq 2$.*
3. *$UC_{qa}(n, n) = UC_{qc}(n, n)$ for all $n \geq 2$.*
4. *$C_{qa}(n, m) = C_{qc}(n, m)$ for all $n, m \geq 2$.*
5. *$C_{qa}(n, 2) = C_{qc}(n, 2)$ for all $n \geq 2$.*

Proof. Suppose (1) holds. By Theorem 1.3.7, Kirchberg's conjecture is equivalent to every $(\min, \text{er})$-nuclear operator system being (el, c)-nuclear. As each $\mathcal{V}_n$ is $(\min, \text{er})$-nuclear, it follows that $\mathcal{V}_{n_1} \otimes_{\min} \mathcal{V}_{n_2} = \mathcal{V}_{n_1} \otimes_c \mathcal{V}_{n_2}$ for all $n_1, n_2 \geq 2$. Hence, these operator systems are order isomorphic, so that $UC_{qa}(n_1, n_2) = UC_{qc}(n_1, n_2)$ for all $n_1, n_2 \geq 2$. Clearly (2) implies (3) and (4) implies (5). The implication (5) $\implies$ (1) holds by Theorem 1.9.5. Hence, we need only show that (3) implies (5). By Lemma 2.3.11, condition (3) implies that $\mathcal{V}_n \otimes_{\min} \mathcal{V}_n$ and $\mathcal{V}_n \otimes_c \mathcal{V}_n$ are order isomorphic. By Proposition 2.3.4, $NC(n)$ is a retract of $\mathcal{V}_n$. Using Lemma 2.1.10, the identity map $\text{id} : NC(n) \otimes_{\min} NC(n) \to NC(n) \otimes_c NC(n)$ is 1-positive. Since $\min \leq c$, we see that $NC(n) \otimes_{\min} NC(n)$ and $NC(n) \otimes_c NC(n)$ are order isomorphic for all $n \geq 2$. Applying Proposition 2.3.3, we obtain the equality $C_{qa}(n, 2) = C_{qc}(n, 2)$, as desired. $\square$

Some striking differences arise between the quantum correlation sets and the unitary correlation sets. It is known that $C_{qa}(2,2) = C_{qc}(2,2)$ (see, for example, [47]). The question of whether $C_q(n,m) = C_{qc}(n,m)$ for all $n,m \geq 2$ was open until Slofstra [59] recently proved that there are large n,m for which $C_{qs}(n,m) \neq C_{qc}(n,m)$. Similarly, it was unknown whether $C_{qs}(n,m)$ is closed for all $n,m \geq 2$, until Slofstra recently provided a counterexample [60] for large n,m. The counterexample with the smallest known input and output sets is due to K. Dykema, V. Paulsen and J. Prakash, who proved that $C_{qs}(5,2) \neq C_{qa}(5,2)$ [20].

In contrast, we will see in Chapter 3 that $UC_{qs}(2,2) \subsetneq UC_{qc}(2,2)$. Indeed, in [11] it is shown that there is a state $s : \mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2) \to \mathbb{C}$ that cannot arise from a finite-dimensional representation of $\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2)$. In fact, it is shown that this state cannot arise from a spatial representation of $\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2)$ on a tensor product of Hilbert spaces, even if the Hilbert spaces are infinite-dimensional. Since $C_n^{\text{comm}}(\mathcal{U}_{nc}(2), \mathcal{U}_{nc}(2)) \subseteq C_n^{\min}(\mathcal{U}_{nc}(2), \mathcal{U}_{nc}(2))$, s is also a state on $\mathcal{U}_{nc}(2) \otimes_{\max} \mathcal{U}_{nc}(2)$. Hence we obtain an element of $UC_{qc}(2,2)$ that cannot be in $UC_{qs}(2,2)$. This will be used to show that $UC_{qs}(2,2) \subsetneq UC_{qc}(2,2)$. Moreover, it is shown in [11] that s can be approximated in the w^*-topology by states on $\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2)$ corresponding to elements of $UC_q(2,2)$. It will follow that $UC_q(2,2)$ and $UC_{qs}(2,2)$ are not even closed. The methods in [11] will be adapted in a natural way in Chapter 3 to show that $UC_{qs}(n_1,n_2) \subsetneq UC_{qc}(n_1,n_2)$ for all $n \geq 2$, and that $UC_q(n_1,n_2)$ and $UC_{qs}(n_1,n_2)$ are not closed.

Chapter 3

Unitary correlation sets

In this chapter, we will consider a smaller version of the unitary correlation sets from Chapter 2. Our first main result is Theorem 3.2.7, which states that Connes' embedding problem is equivalent to deciding whether a certain compression $B_{qc}(n,n)$ of $UC_{qc}(n,n)$ is equal to the closure of the analogous compression $B_q(n,n)$ of $UC_q(n,n)$. We show that $\overline{B_q(n,n)}$ and $B_{qc}(n,n)$ are the unit balls of certain cross norms on $M_n \otimes M_n$, and that the embedding problem is equivalent to determining whether or not these cross norms are equal on $M_n \otimes M_n$, for all $n \geq 2$. Drawing on the phenomenon of embezzling entanglement, we will show that $B_q(n,m) \neq B_{qc}(n,m)$ for all $n,m \geq 2$ and that $B_q(n,m)$ is not closed. This result is one way in which the unitary correlation sets differ greatly from the quantum bipartite correlation sets.

Section 3.1 gives some properties of the smaller unitary correlation sets $B_q(n,m)$ and $B_{qc}(n,m)$, along with other related unitary correlation sets. Moreover, the correspondence between these correlation sets and cross norms on $M_n \otimes M_m$ is given. We relate Connes' embedding problem to determining whether or not $\overline{B_q(n,n)} = B_{qc}(n,n)$ in Section 3.2. Finally, in Section 3.3, we use the theory of embezzling entanglement of states from [11] and [64] to demonstrate several separations between the various unitary correlation sets.

3.1 Unitary correlation norms

In this section, we consider a smaller subset of the correlation sets $UC_t(n_1,n_2)$ from the previous chapter, and show that they define certain cross-norms on $M_{n_1} \otimes M_{n_2}$. We recall that, by Theorem 2.1.3, the map $\varphi_n : M_{2n} \to \mathcal{V}_n$ defined in the previous chapter is a complete quotient map. As an immediate corollary, we obtain the following:

Corollary 3.1.1. *If $n, m \in \mathbb{N}$ with $n, m \geq 2$, then $\varphi_n \otimes \varphi_m : M_{2n} \otimes M_{2m} \to \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$ is a complete quotient map.*

Proof. It is straightforward to check that if $\mathcal{J}_{2n}$ is the kernel of φ_n, then

$$\ker(\varphi_n \otimes \varphi_m) = \mathcal{J}_{2n} \otimes M_{2m} + M_{2n} \otimes \mathcal{J}_{2m}.$$

Now, let $X \in M_p(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ be strictly positive; that is, assume that $X \geq \varepsilon 1$ for some $\varepsilon > 0$. Then there are $S \in M_k(\mathcal{V}_n)_+$, $T \in M_q(\mathcal{V}_m)_+$, and a rectangular matrix $A \in M_{p,kq}$ such that

$$X = A(S \otimes T)A^*.$$

Since φ_n and φ_m are complete quotient maps and $\mathcal{J}_{2n}$ and $\mathcal{J}_{2m}$ are completely order proximinal, we may find matrices P, Q with entries in M_{2n} and M_{2m}, respectively, with quotient images equal to S and T respectively. Then X is the image of a positive element in $M_{2n} \otimes M_{2m}$, and we are done. $\square$

We will explore properties of the unitary correlation sets defined in Chapter 2, while adding two models: the local model and the maximal model. Recall that, whenever $\mathcal{H}$ is a Hilbert space and $U = (U_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ is unitary, we have defined the set $\mathfrak{B}_n(U) = \{I_{\mathcal{H}}\} \cup \{U_{ij}, U_{ij}^*\}_{i,j=1}^n$. For local correlations, we let $UC_{loc}(n, m)$ be the set of correlations in $UC_{qc}(n, m)$ of the form

$$\{(\langle XY, \psi, \psi \rangle)_{X \in \mathfrak{B}_n(U), Y \in \mathfrak{B}_m(V)}\},$$

where $C^*(\mathfrak{B}_n(U) \cup \mathfrak{B}_m(V))$ is a commutative C^*-algebra.

For each of the correlation sets $UC_t(n, m)$ with $t \in \{loc, q, qs, qa, qc\}$, we will consider the smaller set $B_t(n, m)$ obtained by only considering $X \in \{U_{ij} : 1 \leq i, j \leq n\}$ and $Y \in \{V_{k\ell} : 1 \leq k, \ell \leq m\}$.

To define quantum maximal unitary correlation sets, we will require a slightly different approach. We let

$$\mathcal{G}_{n,m} = \{x \otimes y : x \in \{1\} \cup \{u_{ij}, u_{ij}^*\}_{i,j=1}^n,\ y \in \{1\} \cup \{v_{k\ell}, v_{k\ell}^*\}_{k,\ell=1}^m\}.$$

We let $UC_{qmax}(n, m)$ be the set of all coordinates of the form

$$(s(x))_{x \in \mathcal{G}_{n,m}},$$

where s is a state on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. We let

$$B_{qmax}(n, m) = \{(s(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)} : s \in \mathcal{S}(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)\}.$$

Similarly, we define

$$B_{qmin}(n,m) = \{(s(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)} : s \in \mathcal{S}(\mathcal{V}_n \otimes_{\min} \mathcal{V}_m)\}.$$

Some of the known properties of these sets are summarized in the following theorem. Aside from the presence of $UC_{qmax}(n,m)$, this is the same as Theorem 2.3.9.

Theorem 3.1.2. *Let $n, m \geq 2$. Then*

$$UC_q(n,m) \subseteq UC_{qs}(n,m) \subseteq UC_{qmin}(n,m) \subseteq UC_{qc}(n,m) \subseteq UC_{qmax}(n,m),$$

and each of these sets is convex. Moreover, $\overline{UC_q(n,m)} = \overline{UC_{qs}(n,m)} = UC_{qmin}(n,m)$ and $UC_{qc}(n,m)$ are closed.

Proof. The containment $UC_{qc}(n,m) \subseteq UC_{qmax}(n,m)$ is the only result not shown in Theorem 2.3.9. To show this containment, we use the fact that $UC_{qc}(n,m)$ corresponds to states on $\mathcal{V}_n \otimes_c \mathcal{V}_m$, while $UC_{qmax}(n,m)$ corresponds to states on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. As every state on $\mathcal{V}_n \otimes_c \mathcal{V}_m$ is a state on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$, we obtain the desired inclusion. Since $UC_{qmax}(n,m)$ corresponds to a state space, it is clearly convex, as required. $\square$

A simple but crucial observation is that for $t_1, t_2 \in \{loc, qa, qc, qmax\}$, if $UC_{t_1}(n,m) = UC_{t_2}(n,m)$, then $B_{t_1}(n,m) = B_{t_2}(n,m)$. Hence, one way to separate $UC_{t_1}(n,m)$ and $UC_{t_2}(n,m)$ is by separating the sets $B_{t_1}(n,m)$ and $B_{t_2}(n,m)$. We will see that, for Connes' embedding problem, it suffices to consider the sets $B_t(n,m)$. Moreover, the sets $B_t(n,m)$ for $t \in \{loc, qa, qc, qmax\}$ have a very special structure, as seen below.

Theorem 3.1.3. *For $t \in \{loc, qa, qc, qmax\}$, the set $B_t(n,m)$ is the unit ball of a norm $\|\cdot\|_t$ on $M_n \otimes M_m$. Moreover, $\|\cdot\|_{loc}$ is the norm arising from the projective Banach space tensor product $M_n \otimes_\pi M_m$.*

Proof. As each set $B_t(n,m)$ corresponds to images of states, it is easy to see that $B_t(n,m)$ is convex. Since 0 is a contraction in M_n, there is a state $\eta_n : \mathcal{V}_n \to \mathbb{C}$ with $\eta(u_{ij}) = 0$ for all i, j. By functoriality of the min tensor product, $\eta_n \otimes \eta_m : \mathcal{V}_n \otimes_{\min} \mathcal{V}_m \to \mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ is a state, which corresponds to the matrix $0 \in M_{nm}$. Each entry of a matrix in $B_t(n,m)$ must have modulus at most 1, so the set $B_t(n,m)$ is clearly compact in M_{nm}. It remains to show that 0 is an interior point in $B_t(n,m)$. Since $B_{loc}(n,m)$ is the smallest of the correlation sets, it suffices to prove that 0 is an interior point for $B_{loc}(n,m)$. Since $\mathbb{C}$ is a commutative C^*-algebra, any pair of unitary matrices $X \in M_n$ and $Y \in M_m$ satisfies $X \otimes Y \in B_{loc}(n,m)$. Using the fact that the convex hull of the unitaries in M_n is the unit ball of the operator norm in M_n, we see that $\{X \otimes Y \in M_n \otimes M_m : \|X\|, \|Y\| \leq 1\} \subseteq B_{loc}(n,m)$. By Proposition 1.2.2, the closed convex hull of the former set is the unit ball of the projective Banach space tensor product norm; hence, it follows that 0 is an interior point for $B_{loc}(n,m)$. Therefore, each $B_t(n,m)$ is the unit ball of a norm $\|\cdot\|_t$ on M_{nm}.

It remains to show that $\|\cdot\|_{loc} = \|\cdot\|_{\pi}$. To this end, let $\mathcal{A}$ be a unital, commutative C^*-algebra, and let $U \in M_n(\mathcal{A})$ and $V \in M_m(\mathcal{A})$ be unitary. Note that $\mathcal{A} \simeq C(X)$ for some compact Hausdorff space X, so that the extreme points of $\mathcal{S}(\mathcal{A})$ are just the evalation functionals $\{\delta_x : x \in X\}$. The matrix in $B_{loc}(n,m)$ arising from one of these states corresponding to U and V is $(\delta_x(u_{ij}v_{k\ell})) = (\delta_x(u_{ij})\delta_x(v_{k\ell}))$. Note that $(\delta_x(u_{ij}))$ and $(\delta_v(v_{k\ell}))$ are contractions in M_n and M_m respectively, so that $(\delta_x(u_{ij})\delta_x(v_{k\ell}))$ is of the form $A \otimes B$ where $A \in M_n$ and $B \in M_m$ are contractions. Taking the closed convex hull of the pure states on $C(X)$, we see that every element of $B_{loc}(n,m)$ is in the closed convex hull of $\{A \otimes B : A \in M_n, \, B \in M_m, \, \|A\| \leq 1, \, \|B\| \leq 1\}$. This shows that $\|\cdot\|_{loc}$ is the projective Banach space tensor norm on $M_n \otimes M_m$, as desired. □

We will see later that if $t \neq qmax$, then $\|\cdot\|_t$ cannot be unitarily invariant. However, all of these norms satisfy a weaker condition.

Proposition 3.1.4. *For $t \in \{loc, qa, qc\}$, the norm $\|\cdot\|_t$ is locally unitarily invariant on $M_n \otimes M_m$; i.e., for any unitaries $U_1, U_2 \in M_n$, unitaries $V_1, V_2 \in M_m$ and $X \in M_n \otimes M_m$, we have*

$$\|(U_1 \otimes V_1)X(U_2 \otimes V_2)\|_t = \|X\|_t.$$

Proof. First, let s be a state on $\mathcal{V}_n \otimes_c \mathcal{V}_m$. Then there is a Hilbert space $\mathcal{H}$, unitaries $U = (U_{ij}) \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V = (V_{k\ell}) \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^m)$, and a unit vector $\psi \in \mathcal{H}$ such that $s(u_{ij} \otimes v_{k\ell}) = \langle U_{ij}V_{k\ell}\psi, \psi\rangle$. Let $X = (s(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)} \in M_n \otimes M_m$. We will show that $X[(\alpha_{ij}) \otimes I] \in UC_{qc}(n,m)$ whenever (α_{ij}) is a unitary matrix in M_n; the rest of the cases will follow. Define $\widehat{U}_{ij} = \sum_{p=1}^n U_{ip}\alpha_{pj}$. Then $(\widehat{U}_{ij}) = U(\alpha_{ij})$ is unitary, and $\widehat{U}_{ij}V_{k\ell} = V_{k\ell}\widehat{U}_{ij}$. It follows that

$$X((\alpha_{ij}) \otimes I) = (\langle U_{ij}V_{k\ell}\psi, \psi\rangle)(\alpha_{ij} \otimes I) = (\langle \widehat{U}_{ij}V_{k\ell}\psi, \psi\rangle) \in UC_{qc}(n,m).$$

If the entries of U and V generate a commutative C^*-algebra, then the same is true for the entries of $\widehat{U} = (\widehat{U}_{ij})$ and V, so that $B_{loc}(n,m)$ is locally unitarily invariant. If we assume that $X \in B_{qs}(n,m)$, then $s(u_{ij} \otimes v_{k\ell})$ can be written as $\langle (U_{ij} \otimes V_{k\ell})\psi, \psi\rangle$, where $U = (U_{ij}) \in M_n(\mathcal{B}(\mathcal{H}_A))$ and $V = (V_{k\ell}) \in M_m(\mathcal{B}(\mathcal{H}_B))$ are unitary, $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces, and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a unit vector. Applying the same approach as above, the matrix $X((\alpha_{ij}) \otimes I)$ arises from a state induced by a tensor product of representations, so that

$X((\alpha_{ij}) \otimes I) \in B_{qs}(n,m)$. Therefore, the set $B_{qs}(n,m)$ is also locally unitarily invariant. The fact that $B_{qa}(n,m)$ is locally unitarily invariant follows by taking limits and using the fact that matrix multiplication is continuous in the norm topology on M_{nm}. $\square$

Like the norm $\|\cdot\|_{loc}$, each of the norms $\|\cdot\|_t$ must be a reasonable cross-norm.

Theorem 3.1.5. *For $t \in \{loc, qa, qc, qmax\}$, $\|\cdot\|_t$ is a reasonable cross-norm on $M_n \otimes M_m$. Moreover, if $\|\cdot\|$ denotes the operator norm on M_{nm}, then $\|\cdot\| \leq \|\cdot\|_t$.*

Proof. Since $B_{loc}(n,m) \subseteq B_t(n,m)$ for $t \in \{qa, qc, qmax\}$, we know that $\|X \otimes Y\|_t \leq 1$ whenever $X \in M_n$ and $Y \in M_m$ satisfy $\|X\| \leq 1$ and $\|Y\| \leq 1$. Hence, $\|\cdot\|_t$ is a cross-norm. Once we show that $\|\cdot\| \leq \|\cdot\|_t$, we will have $\|\cdot\|_\varepsilon \leq \|\cdot\|_t \leq \|\cdot\|_\pi$, where $\|\cdot\|_\varepsilon$ is the injective Banach space tensor norm, which shows that $\|\cdot\|_t$ is a reasonable cross-norm. To see that $\|\cdot\|_t \geq \|\cdot\|$, we need only show that $\|\cdot\| \leq \|\cdot\|_{qmax}$, since $\|\cdot\|_{qmax}$ defines the smallest $\|\cdot\|_t$. Let $X \in B_{qmax}(n,m)$; then there is a state $s \in \mathcal{S}(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$ with $X = (s(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)}$. Any operator system tensor product is an operator space tensor product by Proposition 1.2.13. Since $\|(u_{ij})\| = 1$ and $\|(v_{k\ell})\| = 1$, we must have $\|(u_{ij} \otimes v_{k\ell})\| = 1$ in $M_{nm}(\mathcal{V}_n \otimes_{\max} \mathcal{V}_m)$. Since s is completely contractive, we see that $\|X\| \leq 1$ in M_{nm}, and the result follows. $\square$

The lower bound in Theorem 3.1.5 is attained by the norm arising from $B_{qmax}(n,m)$.

Theorem 3.1.6. *For $n, m \geq 2$, the norm $\|\cdot\|_{qmax}$ with unit ball equal to $B_{qmax}(n,m)$ is the operator norm on M_{nm}. In other words,*

$$B_{qmax}(n,m) = \{X \in M_{nm} : \|X\| \leq 1\}.$$

Proof. Theorem 3.1.5 shows that $B_{qmax}(n,m) \subseteq \{X \in M_{nm} : \|X\| \leq 1\}$. For the reverse inclusion, let $X \in M_n \otimes M_m$ with operator norm at most 1. We may write

$$X = \sum_{\substack{1 \leq i,j \leq n \\ 1 \leq k,\ell \leq m}} x_{ijk\ell} E_{ij} \otimes E_{k\ell}.$$

Define the element

$$\chi = \sum_{i,j,k,\ell} x_{ijk\ell} \otimes E_{12} \otimes E_{ij} \otimes E_{12} \otimes E_{k\ell} \in M_2 \otimes M_n \otimes M_2 \otimes M_m,$$

and let

$$P = I_2 \otimes I_n \otimes I_2 \otimes I_m + \chi + \chi^*.$$

Since $\|\chi\| \le 1$ in $M_{2n} \otimes M_{2m}$, P is positive in $M_{2n} \otimes M_{2m}$. Therefore, the corresponding map $\gamma_P : M_2 \otimes M_n \otimes M_2 \otimes M_m \to \mathbb{C}$ with Choi matrix equal to P is a positive linear functional; moreover, $\gamma_P(I_2 \otimes I_n \otimes I_2 \otimes I_m) = 4mn$.

Let $\mathcal{J}_{2n} = \ker \varphi_n$, where $\varphi_n : M_{2n} \to \mathcal{V}_n$ is the complete quotient map in Theorem 2.1.3. We claim that $\gamma_P(\mathcal{J}_{2n} \otimes M_{2m} + M_{2n} \otimes \mathcal{J}_{2m}) = 0$, so that γ_P induces a positive linear functional $\widetilde{\gamma}_P$ on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. To show this, we will show that γ_P annihilates $\mathcal{J}_{2n} \otimes M_{2m}$; the other part is similar. We may write $\mathcal{J}_{2n} \otimes M_{2m}$ as the set of all elements of the form

$$C := (E_{11} \otimes A + E_{22} \otimes B) \otimes W,$$

where $A, B \in M_n$, $W \in M_{2m}$ and $\mathrm{Tr}(A) + \mathrm{Tr}(B) = 0$. Applying γ_P to the element C, we obtain

$$\gamma_P(C) = \mathrm{Tr}(A)\,\mathrm{Tr}(W) + \mathrm{Tr}(B)\,\mathrm{Tr}(W) = (\mathrm{Tr}(A) + \mathrm{Tr}(B))\,\mathrm{Tr}(W) = 0.$$

It follows that $\gamma_P(\ker(\varphi_n \otimes \varphi_m)) = 0$. By Proposition 1.1.4 and Corollary 3.1.1, the induced functional $\widetilde{\gamma}_P : \mathcal{V}_n \otimes_{\max} \mathcal{V}_m \to \mathbb{C}$ is positive with $\widetilde{\gamma}_P(1) = \gamma_P(I_2 \otimes I_n \otimes I_2 \otimes I_m) = 4mn$. Let $s = \frac{1}{4mn}\widetilde{\gamma}_P$, which is a state on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. We observe that

$$s(u_{ij} \otimes v_{k\ell}) = \frac{1}{4mn}\widetilde{\gamma}_P(u_{ij} \otimes v_{k\ell}) = \widetilde{\gamma}_P\left(\frac{1}{2n}u_{ij} \otimes \frac{1}{2m}v_{k\ell}\right).$$

Recall that the quotient image of $E_{12} \otimes E_{ij} \in M_2 \otimes M_n$ under the map φ_n is $\frac{1}{2n}u_{ij}$, and similarly, the quotient image of $E_{12} \otimes E_{k\ell} \in M_2 \otimes M_m$ under the map φ_m is $\frac{1}{2m}v_{k\ell}$. Therefore,

$$s(u_{ij} \otimes v_{k\ell}) = \gamma_P(E_{12} \otimes E_{ij} \otimes E_{12} \otimes E_{k\ell}) = x_{ijk\ell},$$

so that $(s(u_{ij} \otimes v_{k\ell})) = X$. We conclude that $X \in B_{qmax}(n, m)$, as desired. $\square$

A careful examination of Theorem 2.1.3 and the proof of Theorem 3.1.6 shows that for $X \in B_{qmax}(n, m)$, there is a state on $\mathcal{V}_n \otimes_{\max} \mathcal{V}_m$ such that $(s(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)} = X$ and $s(x) = 0$ for every $x \in \mathcal{G}_{n,m} \setminus \{u_{ij} \otimes v_{k\ell}\}_{i,j,k,\ell}$. The following proposition shows that such a state can always be found for elements of $B_t(n, m)$, where $t \in \{loc, qa, qc\}$.

Proposition 3.1.7. *Let $t \in \{loc, qa, qc\}$ and $X \in B_t(n, m)$. Then there is a state s on $\mathcal{V}_n \otimes_c \mathcal{V}_m$ such that $s(u_{ij} \otimes 1) = 0 = s(1 \otimes v_{k\ell})$ and $s(u_{ij} \otimes v_{k\ell}^*) = 0$ for all i, j, k, ℓ, and $(s(u_{ij} \otimes v_{k\ell})) = X$. If $X \in B_{qa}(n, m)$, then s can be taken to be a state on $\mathcal{V}_n \otimes_{\min} \mathcal{V}_m$.*

Proof. Using the containments $B_{loc}(n,m) \subseteq B_{qc}(n,m)$ and $B_{qa}(n,m) \subseteq B_{qc}(n,m)$, there is a state ω on $\mathcal{V}_n \otimes_c \mathcal{V}_m$ with $(\omega(u_{ij} \otimes v_{k\ell})) = X$. Let U_{ij} and $V_{k\ell}$ be operators on a Hilbert space $\mathcal{H}$ and let $\psi \in \mathcal{H}$ be a unit vector such that $U = (U_{ij})$ and $V = (V_{k\ell})$ are unitaries in $M_n(\mathcal{B}(\mathcal{H}))$ and $M_m(\mathcal{B}(\mathcal{H}))$ respectively; $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ for all i,j,k,ℓ; and $\omega(u_{ij} \otimes v_{k\ell}) = \langle U_{ij}V_{k\ell}\psi, \psi\rangle$. For $\theta \in [0, 2\pi]$, define ω_θ to be the state on $\mathcal{V}_n \otimes_c \mathcal{V}_m$ corresponding to the unitaries $U_\theta = (e^{i\theta}U_{ij})$ and $V_\theta = (e^{-i\theta}V_{k\ell})$ and unit vector ψ. Then the entries of U_θ and V_θ still $*$-commute, and $\langle (U_\theta)_{ij}(V_\theta)_{k\ell}\psi, \psi\rangle = X_{(i,j),(k,\ell)}$ for all i,j,k,ℓ. It is immediate that $\omega_\theta(u_{ij} \otimes 1) = e^{i\theta}\omega(u_{ij} \otimes 1)$, $\omega_\theta(1 \otimes v_{k\ell}) = e^{-i\theta}\omega(1 \otimes v_{k\ell})$ and $\omega_\theta(u_{ij} \otimes v_{k\ell}^*) = e^{2i\theta}\omega(u_{ij} \otimes v_{k\ell}^*)$. Define $s : \mathcal{V}_n \otimes \mathcal{V}_m \to \mathbb{C}$ by

$$s(x) = \frac{1}{2\pi}\int_0^{2\pi} \omega_\theta(x)\, d\theta,$$

which defines a state on $\mathcal{V}_n \otimes_c \mathcal{V}_m$, satisfying $s(1 \otimes v_{k\ell}) = 0 = s(u_{ij} \otimes 1)$ and $s(u_{ij} \otimes v_{k\ell}^*) = 0$. If $X \in B_{qa}(n,m)$, then the state s can be taken to be be a limit of states on $\mathcal{V}_n \otimes_{\min} \mathcal{V}_m$, so that s is a state on $\mathcal{V}_n \otimes_{\min} \mathcal{V}_m$. $\square$

3.2 Unitary correlation norms and Connes' embedding problem

We now move towards another equivalent statement of Connes' embedding problem. We will show that the equality of the qa and qc norms on $M_n \otimes M_n$ is equivalent to a positive answer to the embedding problem. First, we adopt some notation. Let $(w_i)_{i=1}^\infty$ be a set of universal generators for $\mathbb{F}_\infty$. We define the following operator systems:

$$\begin{aligned}
\mathcal{X}_n &= \text{span}\,(\{1\} \cup \{w_i \otimes w_j, w_i^* \otimes w_j^*\}_{i,j=1}^n) \subseteq C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n),\\
\mathcal{Y}_n &= \text{span}\,(\{1\} \cup \{w_i \otimes w_j, w_i^* \otimes w_j^*\}_{i,j=1}^n) \subseteq C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n),\\
\mathcal{X}_\infty &= \text{span}\,(\{1\} \cup \{w_i \otimes w_j, w_i^* \otimes w_j^*\}_{i,j=1}^\infty) \subseteq C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) \text{ and}\\
\mathcal{Y}_\infty &= \text{span}\,(\{1\} \cup \{w_i \otimes w_j, w_i^* \otimes w_j^*\}_{i,j=1}^\infty) \subseteq C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty).
\end{aligned}$$

Proposition 3.2.1. *Let $n \geq 2$. If $B_{qa}(n,n) = B_{qc}(n,n)$, then the identity map $id : \mathcal{X}_n \to \mathcal{Y}_n$ is an order isomorphism.*

Proof. Since $B_{qa}(n,n) = B_{qc}(n,n)$, the identity map

$$\text{id} : \text{span}\,(\{1\} \cup \{u_{ij} \otimes v_{k\ell}, u_{ij}^* \otimes v_{k\ell}^*\}_{i,j,k,\ell}) \to \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$$

from $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n)$ is an order isomorphism onto its range. By the proof of Lemma 2.1.11, there are ucp maps $\psi_n : C^*(\mathbb{F}_n) \to \mathcal{U}_{nc}(n)$ and $\pi_n : \mathcal{U}_{nc}(n) \to C^*(\mathbb{F}_n)$ such that $\mathrm{id}_{C^*(\mathbb{F}_n)} = \pi_n \circ \psi_n$. Moreover, $\psi_n(w_i) = u_{ii}$ and $\pi_n(u_{ij}) = \delta_{ij} w_i$. By functoriality of the min and max tensor products, $\psi_n \otimes \psi_n : C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) \to \mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n)$ and $\psi_n \otimes \psi_n : C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \to \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$ are complete order embeddings. Therefore, $\mathcal{X}_n$ is completely order isomorphic to span $(\{1\} \cup \{u_{ii} \otimes v_{jj}, u_{ii}^* \otimes v_{jj}^*\}_{i,j=1}^n)$ inside of $\mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(n)$; the analogous result holds for $\mathcal{Y}_n$ inside of $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(n)$. It follows that $\mathrm{id} : \mathcal{X}_n \to \mathcal{Y}_n$ is an order isomorphism. □

Next, we need a few facts about the opposite algebra of a C^*-algebra. Given a C^*-algebra $\mathcal{A}$, the **opposite algebra** of $\mathcal{A}$, denoted by $\mathcal{A}^{op}$, is a C^*-algebra with the same norm and $*$-vector space structure as $\mathcal{A}$, but with multiplication given by $(a^{op} b^{op}) = (ba)^{op}$. In particular, whenever $\pi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism between C^*-algebras, there is an associated $*$-homomorphism $\pi^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ given by $\pi^{op}(a^{op}) = (\pi(a))^{op}$. A helpful fact is that, for $a, b \in \mathcal{A}$, we have $\|ba\| = \|(ba)^{op}\| = \|a^{op} b^{op}\|$. A special case of the opposite algebra is for $C^*(\mathbb{F}_\infty)$. In this case, the opposite algebra $C^*(\mathbb{F}_\infty)^{op}$ has the universal property that, whenever $U_i \in \mathcal{B}(\mathcal{H})$ are unitary for $i \in \mathbb{N}$, there is a unital $*$-homomorphism $\pi^{op} : C^*(\mathbb{F}_\infty)^{op} \to \mathcal{B}(\mathcal{H})$ such that

$$\pi^{op}(w_{i_1}^{op} \cdots w_{i_n}^{op}) = U_{i_n} \cdots U_{i_1}.$$

The following fact is well-known; we include the proof for convenience.

Proposition 3.2.2. *The mapping $w_i \mapsto (w_i^*)^{op}$ extends to a unital $*$-isomorphism from $C^*(\mathbb{F}_\infty)$ onto $C^*(\mathbb{F}_\infty)^{op}$.*

Proof. Let $(w_i)_{i=1}^\infty$ be a set of universal generators for $C^*(\mathbb{F}_\infty)$. Let $\mathcal{H}$ be a Hilbert space with fixed orthonormal basis $\{e_\alpha\}_{\alpha \in A}$. For any $T \in \mathcal{B}(\mathcal{H})$, we define $\overline{T} \in \mathcal{B}(\mathcal{H})$ by $\langle \overline{T} e_\alpha, e_\beta \rangle = \overline{\langle T e_\alpha, e_\beta \rangle}$ for all $\alpha, \beta \in A$. Then $\overline{T} \in \mathcal{B}(\mathcal{H})$ and $\|\overline{T}\| = \|T\|$. If U_i is unitary in $\mathcal{B}(\mathcal{H})$ for each $i \in \mathbb{N}$, then it is straightforward to check that $\overline{U_i}$ is unitary and $\overline{U_{i_1}} \cdot \overline{U_{i_2}} = \overline{U_{i_1} U_{i_2}}$ for all i_1, i_2. To ease notation, if $w_\gamma = w_{i_1} \cdots w_{i_n}$ is a reduced word in $\mathbb{F}_\infty$, then we set $U_\gamma = U_{i_1} \cdots U_{i_n}$. Let $w_{\gamma_1}, ..., w_{\gamma_n}$ be reduced words in $\mathbb{F}_\infty$, and let $\lambda_{\gamma_1}, ..., \lambda_{\gamma_n} \in \mathbb{C}$. By the universal property of $C^*(\mathbb{F}_\infty)$, it follows that

$$\left\| \sum_{j=1}^n \lambda_{\gamma_j} w_{\gamma_j} \right\| \geq \left\| \sum_{j=1}^n \lambda_{\gamma_j} \overline{U}_{\gamma_j} \right\|.$$

We observe that

$$\overline{\sum_{j=1}^n \lambda_{\gamma_j} \overline{U}_{\gamma_j}} = \sum_{j=1}^n \overline{\lambda}_{\gamma_j} U_{\gamma_j},$$

so that

$$\left\| \sum_{j=1}^{n} \lambda_{\gamma_j} \overline{U}_{\gamma_j} \right\| = \left\| \sum_{j=1}^{n} \overline{\lambda}_{\gamma_j} U_{\gamma_j} \right\|.$$

Taking the supremum over all representations of $C^*(\mathbb{F}_\infty)$, it follows that

$$\left\| \sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j} \right\| \geq \left\| \sum_{j=1}^{n} \overline{\lambda}_{\gamma_j} w_{\gamma_j} \right\|.$$

A symmetric argument yields the reverse inequality, so that

$$\left\| \sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j} \right\| = \left\| \sum_{j=1}^{n} \overline{\lambda}_{\gamma_j} w_{\gamma_j} \right\|.$$

Since $((w_{i_1} \cdots w_{i_n})^*)^{op} = (w_{i_n}^* \cdots w_{i_1}^*)^{op} = (w_{i_1}^*)^{op} \cdots (w_{i_n}^*)^{op}$, the mapping $\Phi : w_i \mapsto (w_i^*)^{op}$ extends to a unital $*$-homomorphism from the $*$-algebra generated by $(w_i)_{i=1}^{\infty}$ into $C^*(\mathbb{F}_\infty)^{op}$.

Using the norm and multiplication structure of the opposite algebra, we obtain

$$\begin{aligned}
\left\| \Phi\left(\sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j} \right) \right\| &= \left\| \sum_{j=1}^{n} \lambda_{\gamma_j} (w_{\gamma_j}^*)^{op} \right\| \\
&= \left\| \left(\sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j}^* \right)^{op} \right\| \\
&= \left\| \sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j}^* \right\|.
\end{aligned}$$

Taking the adjoint of the last expression gives

$$\left\| \Phi\left(\sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j} \right) \right\| = \left\| \sum_{j=1}^{n} \overline{\lambda}_{\gamma_j} w_{\gamma_j} \right\| = \left\| \sum_{j=1}^{n} \lambda_{\gamma_j} w_{\gamma_j} \right\|.$$

Hence, Φ is an isometry on a dense subset of $C^*(\mathbb{F}_\infty)$. It follows that Φ extends to an isometric unital $*$-homomorphism from $C^*(\mathbb{F}_\infty)$ to $C^*(\mathbb{F}_\infty)^{op}$. Clearly it is surjective since $((w_i^*)^{op})_{i=1}^{\infty}$ generates $C^*(\mathbb{F}_\infty)^{op}$, so we obtain the desired isomorphism. □

We require a few results from [47] to relate Connes' embedding problem to states on $\mathcal{X}_\infty$ and $\mathcal{Y}_\infty$.

Theorem 3.2.3. (Ozawa, [47]) *Let $\mathcal{A}$ be a unital C^*-algebra, and let τ be a tracial state on $\mathcal{A}$. Then the map $s_\tau : \mathcal{A} \otimes_{\max} \mathcal{A}^{op} \to \mathbb{C}$ given by $s_\tau(a \otimes b^{op}) = \tau(ab)$ extends to a state on $\mathcal{A} \otimes_{\max} \mathcal{A}^{op}$.*

We obtain the following description of traces in terms of certain states on $\mathcal{Y}_\infty$.

Theorem 3.2.4. (Ozawa, [47]) *Let $\mathcal{A}$ be a separable C^*-algebra, and let $(u_i)_{i=1}^\infty$ be a generating sequence of unitaries in the unitary group of $\mathcal{A}$. Suppose that τ is a tracial state on $\mathcal{A}$. Then the mapping $w_i \otimes w_j \mapsto \tau(u_i u_j^*)$ extends to a state on $\mathcal{Y}_\infty$.*

Proof. We let $\sigma : C^*(\mathbb{F}_\infty) \to \mathcal{A}$ be the surjective unital $*$-homomorphism given by $\sigma(w_i) = u_i$. By Theorem 3.2.3, τ induces a state s_τ on $\mathcal{A} \otimes_{\max} \mathcal{A}^{op}$ given by $a \otimes b^{op} \mapsto \tau(ab)$. Let $\sigma^{op} : (C^*(\mathbb{F}_\infty))^{op} \to \mathcal{A}^{op}$ denote the opposite representation of σ, given by $\sigma^{op}(w_i^{op}) = u_i^{op}$. Then $\sigma \otimes \sigma^{op} : C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \to \mathcal{A} \otimes_{\max} \mathcal{A}^{op}$ is a $*$-homomorphism. Using the fact that $C^*(\mathbb{F}_\infty)^{op} \simeq C^*(\mathbb{F}_\infty)$, we see that the mapping $w_i \otimes w_j \mapsto \tau(u_i u_j^*) = s_\tau \circ (\sigma \otimes \sigma^{op})(w_i \otimes (w_j^*)^{op})$ extends to a state on $\mathcal{Y}_\infty$. $\square$

The key result that links $\mathcal{X}_\infty$ to Connes' embedding problem is the following.

Theorem 3.2.5. (Ozawa, [47]) *Let $\mathcal{A}$ be a separable C^*-algebra with a countable dense sequence $(u_i)_{i=1}^\infty$ of unitaries and a tracial state τ. Then $(\pi_\tau(\mathcal{A})'', \tau)$ satisfies Connes' embedding problem if and only if the mapping $w_i \otimes w_j \mapsto \tau(u_i u_j^*)$ extends to a state on $\mathcal{X}_\infty$.*

In order to use Theorem 3.2.5, we must ensure that each $\mathcal{X}_n$ and $\mathcal{Y}_n$ can be considered inside of the respective tensor product of $C^*(\mathbb{F}_\infty)$.

Lemma 3.2.6. *For each $n \geq 2$, the identity maps $id : \mathcal{X}_n \to \mathcal{X}_\infty$ and $id : \mathcal{Y}_n \to \mathcal{Y}_\infty$ are complete order embeddings.*

Proof. Since the minimal operator system tensor product is injective and $\mathcal{X}_n \subseteq C^*(\mathbb{F}_n) \otimes_{\min} C^*(\mathbb{F}_n) \subseteq C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty)$, the result immediately follows for $\mathcal{X}_n$. Now, the canonical embedding $\mathbb{F}_n \hookrightarrow \mathbb{F}_\infty$ and canonical quotient map $\mathbb{F}_\infty \to \mathbb{F}_n$ give rise to $*$-homomorphisms $\pi_n : C^*(\mathbb{F}_n) \to C^*(\mathbb{F}_\infty)$ and $\sigma_n : C^*(\mathbb{F}_\infty) \to C^*(\mathbb{F}_n)$ with $\sigma_n \circ \pi_n = \mathrm{id}_{C^*(\mathbb{F}_n)}$. By functoriality of the maximal tensor product, $\pi_n \otimes \pi_n$ and $\sigma_n \otimes \sigma_n$ are ucp with respect to the maximal tensor product. Therefore, the following diagram commutes:

$$\begin{tikzcd}
 & C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty) \arrow[dr, "\sigma_n \otimes \sigma_n"] & \\
C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n) \arrow[ur, "\pi_n \otimes \pi_n"] \arrow[rr, "\mathrm{id}"] & & C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n)
\end{tikzcd}$$

Hence, $C^*(\mathbb{F}_n) \otimes_{\max} C^*(\mathbb{F}_n)$ is completely order isomorphic to the image of $C^*(\mathbb{F}_n) \otimes C^*(\mathbb{F}_n)$ in $C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$. Restricting to $\mathcal{Y}_n$ shows that the identity map $\mathrm{id} : \mathcal{Y}_n \to \mathcal{Y}_\infty$ is a complete order isomorphism onto its range. □

We are now ready for the main result of this section.

Theorem 3.2.7. *The following are equivalent.*

1. *Connes' embedding problem has a positive answer.*
2. $UC_{qa}(n,m) = UC_{qc}(n,m)$ *for all* $n, m \geq 2$.
3. $B_{qa}(n,m) = B_{qc}(n,m)$ *for all* $n, m \geq 2$.
4. $B_{qa}(n,n) = B_{qc}(n,n)$ *for all* $n \geq 2$.
5. $M_n \otimes_{qa} M_n = M_n \otimes_{qc} M_n$ *isometrically for all* $n \geq 2$.

Proof. The equivalence of (1) and (2) is by Theorem 2.3.12. Clearly (2) implies (3) and (3) implies (4). Two norms on the same space are equal if and only if their closed unit balls are identical, so (4) is equivalent to (5). Hence, it remains to show that (4) implies (1).

Suppose that $B_{qa}(n,n) = B_{qc}(n,n)$ for all $n \geq 2$. By Proposition 3.2.1, the identity map $\mathrm{id} : \mathcal{X}_n \to \mathcal{Y}_n$ is an order isomorphism for all $n \geq 2$. Let $\mathcal{A}$ be a separable C^*-algebra with a countable dense sequence $(u_i)_{i=1}^\infty$ of unitaries, and let τ be a tracial state on $\mathcal{A}$. By Theorem 3.2.4, the mapping $w_i \otimes w_j \mapsto \tau(u_i u_j^*)$ extends to a state on $\mathcal{Y}_\infty$.

Consider the C^*-algebra $\mathcal{A}_n = C^*(u_1, ..., u_n)$, which has a generating sequence of unitaries given by $(v_i)_{i=1}^\infty$, where $v_i = u_i$ for $i \leq n$ and $v_i = 1$ for $i > n$. Define $s_n : \mathcal{Y}_\infty \to \mathbb{C}$ to be the unital, self-adjoint mapping given by $w_i \otimes w_j \mapsto \tau(v_i v_j^*)$. Then s_n is a state by Theorem 3.2.4. Restricting to $\mathcal{Y}_n$, $(s_n)_{|\mathcal{Y}_n}$ must be a state on $\mathcal{X}_n$. By the Hahn-Banach theorem, we may extend $(s_n)_{|\mathcal{Y}_n}$ to a state on $\mathcal{X}_\infty$, which we will denote by ω_n. If $x \in \mathcal{X}_\infty$, then $x = \lambda 1 + \sum_{i,j=1}^n (\lambda_{ij} w_i \otimes w_j + \mu_{ij} w_i^* \otimes w_j^*)$ for some n, so that $x \in \mathcal{X}_n$. It follows that $\lim_{m \to \infty} \omega_m(x) = \omega_n(x)$. Hence, $(\omega_m)_{m=1}^\infty$ converges pointwise to the linear map $\omega : \mathcal{X}_\infty \to \mathbb{C}$ given by $\omega(1) = 1$, $\omega(w_i \otimes w_j) = \tau(u_i u_j^*)$ and $\omega(w_i^* \otimes w_j^*) = \overline{\tau(u_i u_j^*)}$. Since the state space of $\mathcal{X}_\infty$ is w^*-closed, ω is a state. Therefore, the mapping $w_i \otimes w_j \mapsto \tau(u_i u_j^*)$ extends to a state on $\mathcal{X}_\infty$, so that $(\pi_\tau(\mathcal{A})'', \tau)$ satisfies Connes' embedding problem. Since $\mathcal{A}$ was an arbitrary C^*-algebra with separable unitary group, we see that Connes' embedding problem must have a positive answer. Hence, (4) implies (1). □

3.3 Separating the unitary correlation sets

In this section, we will use results from [11] to show that $B_{qs}(n,m) \neq B_{qc}(n,m)$ for all $n,m \geq 2$; moreover, we will show that $B_{qs}(n,m)$ is not closed. Attempts to obtain comparable results for the probabilistic quantum correlation sets given in Tsirelson's problem have a long history and are less definitive. It was only recently shown by W. Slofstra [59] that there are $n,m \in \mathbb{N}$ such that $C_{qs}(n,m) \neq C_{qc}(n,m)$, but for which pairs these sets are not equal is unknown. Slofstra also showed that there exist n_1, n_2, k_1, k_2 for which the set $C_{qs}(n_1,n_2,k_1,k_2)$ is not closed, where n_1 is the number of inputs for Alice, n_2 is the number of inputs for Bob, k_1 is the number of outputs for Alice, and k_2 is the number of outputs for Bob [60]. (Slofstra's counterexample has $n_1 = 184$, $n_2 = 235$, $k_1 = 8$ and $k_2 = 2$.) The smallest n_1, n_2, k_1, k_2 known for which $C_{qs} \neq C_{qa}$ are $n_1 = n_2 = 5$ and $k_1 = k_2 = 2$. This choice of n_1, n_2, k_1, k_2 is also the smallest known for which $C_{qs} \neq C_{qc}$. In contrast, the two analogous problems for unitary correlation sets have a negative answer for every $n,m \geq 2$, as we will see below.

Before we establish separations between some of the various unitary correlation sets, we require some terminology involving state embezzlement, as described in [11]. We give a somewhat simplified embezzlement framework here. Suppose that Alice and Bob each have access to a finite-dimensional Hilbert space; we will always assume that Alice's space is $\mathbb{C}^n$ and Bob's space is $\mathbb{C}^m$ for some $n,m \geq 2$. Suppose that Alice and Bob have access to a resource Hilbert space $\mathcal{R}$, and are able to act on the system $\mathbb{C}^n \otimes \mathcal{R} \otimes \mathbb{C}^m$ locally. We consider whether there is a unit vector $\psi \in \mathcal{R}$ such that Alice and Bob's operations can send $e_1 \otimes \psi \otimes e_1$ to $\sum_{i,j} \alpha_{ij} e_i \otimes \psi \otimes e_j$, where $\sum_{i,j} |\alpha_{ij}|^2 = 1$. We will say that there is a **perfect embezzlement protocol in a finite-dimensional tensor product model** for $\sum_{i,j} \alpha_{ij} e_i \otimes e_j$ if there is a resource Hilbert space $\mathcal{R} = \mathcal{R}_A \otimes \mathcal{R}_B$, operators $U_{ij} \in \mathcal{B}(\mathcal{R}_A)$ and $V_{k\ell} \in \mathcal{B}(\mathcal{R}_B)$ for $1 \leq i,j \leq n$ and $1 \leq k,\ell \leq m$ such that $U = (U_{ij})$ and $V = (V_{k\ell})$ are unitary on $\mathbb{C}^n \otimes \mathcal{R}_A$ and $\mathcal{R}_B \otimes \mathbb{C}^m$ respectively, with

$$(U \otimes V)(e_1 \otimes \psi \otimes e_1) = \sum_{i,j} \alpha_{ij} e_i \otimes \psi \otimes e_j.$$

We will say that there is a **perfect embezzlement protocol in a tensor product model** for $\sum_{i,j} \alpha_{ij} e_i \otimes e_j$ if the same conditions are met as above, except that we drop the requirement that $\dim(\mathcal{R}_A), \dim(\mathcal{R}_B) < \infty$. A **perfect embezzlement protocol in the commuting model** for $\sum_{i,j} \alpha_{ij} e_i \otimes e_j$ will have the same properties as above, except that we drop the assumption that $\mathcal{R}$ decomposes as a tensor product, and instead assume that $U_{ij}, V_{k\ell} \in \mathcal{B}(\mathcal{R})$ for all i,j,k,ℓ, and that

$$(U \otimes I_m)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_m).$$

The next two results relate perfect embezzlement and states on tensor products of $\mathcal{V}_n$.

Proposition 3.3.1. (Cleve-Liu-Paulsen, [11]) *Let $U_{ij}, V_{k\ell} \in \mathcal{B}(\mathcal{R})$ for $1 \leq i,j \leq n$ and $1 \leq k,\ell \leq m$ be such that $U = (U_{ij})$ and $V = (V_{k\ell})$ are unitary. Then $(U \otimes I_m)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_m)$ if and only if $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ and $U_{ij}^* V_{k\ell} = V_{k\ell}U_{ij}^*$ for all i,j,k,ℓ.*

The following is a slight extension of a result from [11].

Proposition 3.3.2. (Cleve-Liu-Paulsen, [11]) *A perfect embezzlement protocol in the commuting model exists for $\sum_{i,j} \alpha_{ij} e_i \otimes e_j$ if and only if there is a state $s \in \mathcal{S}(\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m))$ such that $s(u_{i1} \otimes v_{j1}) = \alpha_{ij}$.*

Proof. Suppose that $U_{ij}, V_{k\ell} \in \mathcal{B}(\mathcal{R})$ are such that $U = (U_{ij})$ and $V = (V_{k\ell})$ are unitary and $\psi \in \mathcal{R}$ is a unit vector such that $(U \otimes I_m)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_m)$ and $(U \otimes I_m)(I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \sum_{i,j} \alpha_{ij} e_i \otimes \psi \otimes e_j$. Then there is a unital $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m) \to \mathcal{B}(\mathcal{R})$ such that $\pi(u_{ij} \otimes v_{k\ell}) = U_{ij}V_{k\ell}$. Define the state $s : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m) \to \mathbb{C}$ such that $u_{ij} \otimes v_{k\ell} \mapsto \langle U_{ij}V_{k\ell}\psi, \psi\rangle$. By Proposition 2.1.6, $\mathcal{V}_n \otimes_c \mathcal{V}_m$ is completely order isomorphic to its inclusion in $\mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m)$. Hence, we obtain a state $s : \mathcal{V}_n \otimes_c \mathcal{V}_m \to \mathbb{C}$ such that

$$s(u_{i1} \otimes v_{j1}) = \langle U_{i1}V_{j1}\psi, \psi\rangle = \alpha_{ij}.$$

Conversely, suppose that such a state s exists. Then $(s(u_{ij} \otimes v_{k\ell})) \in B_{qc}(n,m)$, so there are unitaries $U = (U_{ij})$ and $V = (V_{k\ell})$ with $U_{ij}, V_{k\ell} \in \mathcal{B}(\mathcal{R})$ and $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$, and a unit vector $\psi \in \mathcal{R}$ such that, for each i,j,k,ℓ, we have $s(u_{ij} \otimes v_{k\ell}) = \langle U_{ij}V_{k\ell}\psi, \psi\rangle$. Now,

$$\begin{aligned} 1 = \sum_{i,j} |\alpha_{ij}|^2 &= \sum_{i,j} |\langle U_{i1}V_{j1}\psi, \psi\rangle|^2 \\ &\leq \sum_{i,j} \|U_{i1}V_{j1}\psi\|^2 = \left\| (U \otimes I_m)(I_n \otimes V) \begin{pmatrix} \psi \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|^2 = 1, \end{aligned}$$

using the fact that $(U \otimes I_m)(I_n \otimes V)$ is unitary. Therefore, $|\langle U_{i1}V_{j1}\psi, \psi\rangle| = \|U_{i1}V_{j1}\psi\|$ for all i,j. Since $\langle U_{i1}V_{j1}\psi, \psi\rangle = \alpha_{ij}$, by the Cauchy-Schwarz inequality, we must have $U_{i1}V_{j1}\psi = \alpha_{ij}\psi$. Therefore, we observe that

$$(U \otimes I_m)(I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \sum_{i,j} \alpha_{ij} e_i \otimes \psi \otimes e_j,$$

so a perfect embezzlement protocol exists in the commuting model for the unit vector $\sum_{i,j} \alpha_{ij} e_i \otimes e_j$. $\square$

We now give a proof that in the commuting model, any norm one vector in $\mathbb{C}^n \otimes \mathbb{C}^m$ can be perfectly embezzled. In particular, we give an alternate proof that any norm one vector in $\mathbb{C}^n \otimes \mathbb{C}^m$ can be approximately embezzled; i.e., one can use unitaries (U_{ij}) and $(V_{k\ell})$ to obtain the mapping $e_1 \otimes \psi \otimes e_1 \mapsto \sum_{i,j} \alpha_{ij} e_i \otimes \psi_\varepsilon \otimes e_j$, where $|\langle \psi, \psi_\varepsilon \rangle| \geq 1 - \varepsilon$ for a small $\varepsilon > 0$. This fact was first proved in [64], and was reproved in [11] for the vector $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$. Our method of proof here draws on a simplification due to Richard Cleve; we kindly thank him for sharing this simplification.

Theorem 3.3.3. *Let $n, m \geq 2$ and let $\sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^m$ have norm 1. There is a state $s \in \mathcal{S}(\mathcal{V}_n \otimes_{\min} \mathcal{V}_m)$ such that $s(u_{i1} \otimes v_{j1}) = \alpha_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. In fact, this state can be taken such that $s(u_{ij} \otimes v_{k\ell}) = 0$ whenever $j \neq 1$ or $\ell \neq 1$.*

Proof. We may reduce to the case when $\alpha_{11} \geq 0$. Indeed, we may choose $z \in \mathbb{T}$ such that $z\alpha_{11} \geq 0$. Then we can first find $s' \in \mathcal{S}(\mathcal{V}_n \otimes_{\min} \mathcal{V}_m)$ such that $s'(u_{i1} \otimes v_{j1}) = z\alpha_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. As the matrix $(\overline{z} u_{ij})$ is also unitary, the map $s : \mathcal{V}_n \otimes_{\min} \mathcal{V}_m \to \mathbb{C}$ given by $s(u_{ij} \otimes v_{k\ell}) = s'(\overline{z} u_{ij} \otimes v_{k\ell}) = \overline{z} s'(u_{ij} \otimes v_{k\ell})$ also extends to a state on $\mathcal{V}_n \otimes_{\min} \mathcal{V}_m$; moreover, $s(u_{i1} \otimes v_{j1}) = \alpha_{ij}$ and $s(u_{ij} \otimes v_{k\ell}) = 0$ whenever $j \neq 1$ or $\ell \neq 1$. Hence, we may assume without loss of generality that $\alpha_{11} \geq 0$.

Let $r \in \mathbb{N}$. Define $h_0 = e_1 \otimes e_1$ and $h_r = \sum_{i,j} \alpha_{ij} e_i \otimes e_j$. Since $\langle h_0, h_r \rangle = \alpha_{11} \geq 0$, it follows that $\mathbb{R}h_0 + \mathbb{R}h_r$ is a two-dimensional real Hilbert space, so there is a unitary $R : \mathbb{R}h_0 + \mathbb{R}h_r \to \mathbb{R}^2$ such that $R(h_0) = e_1$. Since $\|h_0\| = \|h_r\| = 1$, there is an orthogonal matrix $W \in M_2$ such that $We_1 = Rh_r$. It is clear that W must be a rotation of the form $W = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in [0, 2\pi)$. For $1 \leq j \leq r-1$, let $h_j = R^{-1} W_j e_1$, where W_j refers to the rotation $\begin{pmatrix} \cos\left(\frac{j\theta}{r}\right) & -\sin\left(\frac{j\theta}{r}\right) \\ \sin\left(\frac{j\theta}{r}\right) & \cos\left(\frac{j\theta}{r}\right) \end{pmatrix}$. Then $W_p W_q = W_{p+q}$ and $W_p^T = W_{-p}$ for all $p, q \in \mathbb{Z}$, so that, for $1 \leq j \leq r$,

$$\langle h_j, h_{j-1} \rangle = \left\langle R^{-1} W_j e_1, R^{-1} W_{j-1} e_1 \right\rangle = \langle W_1 e_1, e_1 \rangle = \cos\left(\frac{\theta}{r}\right),$$

Let $\psi = h_1 \otimes \cdots \otimes h_r \in (\mathbb{C}^n \otimes \mathbb{C}^m)^{\otimes r}$. Define $U \in \mathcal{B}((\mathbb{C}^n)^{\otimes (r+1)})$ by cyclically shifting the tensors to the right by one position; i.e., for $x_0 \otimes \cdots \otimes x_r \in (\mathbb{C}^n)^{\otimes (r+1)}$, we let

$$U(x_0 \otimes \cdots \otimes x_r) = x_r \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_{r-1}.$$

Then U is unitary and can be identified as a unitary in $M_n(\mathcal{B}((\mathbb{C}^n)^{\otimes r}))$. We define V in the same way on $(\mathbb{C}^m)^{\otimes(r+1)}$. Then $U \otimes V$ is the unitary on $(\mathbb{C}^n \otimes \mathbb{C}^m)^{\otimes(r+1)}$ that permutes the copies of $\mathbb{C}^n \otimes \mathbb{C}^m$ by the cyclic right shift. In particular, we have

$$(U \otimes V)((e_1 \otimes e_1) \otimes \psi) = h_r \otimes \psi_r,$$

where $\psi_r = h_0 \otimes \cdots \otimes h_{r-1}$. In general,

$$(U \otimes V)((e_i \otimes e_j) \otimes \psi) = h_r \otimes (e_i \otimes e_j) \otimes h_1 \otimes \cdots \otimes h_{r-1}.$$

There is a $*$-homomorphism $\pi : \mathcal{U}_{nc}(n) \otimes_{\min} \mathcal{U}_{nc}(m) \to \mathcal{B}((\mathbb{C}^n \otimes \mathbb{C}^m)^{\otimes(r+1)})$ such that

$$(\pi(u_{ij} \otimes v_{k\ell}))_{(i,j),(k,\ell)} = U \otimes V.$$

Define a state $s_r : \mathcal{V}_n \otimes_{\min} \mathcal{V}_m \to \mathbb{C}$ by

$$s_r(x) = \langle \pi(x)\psi, \psi\rangle, \, \forall x \in \mathcal{V}_n \otimes_{\min} \mathcal{V}_m.$$

Then $s_r(u_{i1} \otimes v_{j1}) = \alpha_{ij}\langle \psi, \psi_r\rangle$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We will show that $|\langle \psi, \psi_r\rangle|$ tends to 1 as r becomes large.

It is readily checked that

$$\langle \psi, \psi_r\rangle = \langle h_1, h_0\rangle\langle h_2, h_1\rangle \cdots \langle h_r, h_{r-1}\rangle = \cos\left(\frac{\theta}{r}\right)^r.$$

In particular, $|\langle \psi, \psi_r\rangle|$ tends to 1 as r becomes large. By dropping to a subsequence if necessary, we may assume that $(s_r)_{r=1}^{\infty}$ is a sequence of states converging pointwise. Then $s : \mathcal{V}_n \otimes_{\min} \mathcal{V}_m \to \mathbb{C}$ given by $s(x) = \lim_{r\to\infty} s_r(x)$ is a state on $\mathcal{V}_n \otimes_{\min} \mathcal{V}_m$ such that $s(u_{i1} \otimes v_{j1}) = \alpha_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

It remains to show that $s(u_{ij} \otimes v_{k\ell}) = 0$ whenever $j \neq 1$ or $\ell \neq 1$. Consider the state s_r above, corresponding to the unitaries $U \in \mathcal{B}((\mathbb{C}^n)^{\otimes(r+1)})$ and $V \in \mathcal{B}((\mathbb{C}^m)^{\otimes(r+1)})$ above. Then

$$(U \otimes V)((e_j \otimes e_\ell) \otimes \psi) = h_r \otimes (e_j \otimes e_\ell) \otimes h_1 \otimes \cdots \otimes h_{r-1}.$$

Note that $s_r(u_{ij} \otimes v_{k\ell})$ corresponds to the quantity

$$\langle (U \otimes V)((e_j \otimes e_\ell) \otimes \psi), (e_i \otimes e_k) \otimes \psi\rangle.$$

Therefore,

$$s_r(u_{ij} \otimes v_{k\ell}) = \alpha_{ik}\langle e_j \otimes e_\ell, h_1\rangle\langle h_1, h_2\rangle \cdots \langle h_{r-1}, h_r\rangle.$$

Since $|\langle h_1, h_2\rangle \cdots \langle h_{r-1}, h_r\rangle| \le 1$, we have

$$|s_r(u_{ij} \otimes v_{k\ell})| \le |\alpha_{ik}||\langle e_j \otimes e_\ell, h_1\rangle| \le |\langle e_j \otimes e_\ell, h_1\rangle|.$$

The angle between h_1 and $e_1 \otimes e_1$ is $\frac{\theta}{r}$, so it follows that $\|h_1 - e_1 \otimes e_1\| \to 0$. Thus, $|\langle e_j \otimes e_\ell, h_1\rangle| \to 0$ if $j \neq 1$ or $\ell \neq 1$. This shows that $s_r(u_{ij} \otimes v_{k\ell}) \to 0$ if $j \neq 1$ or $\ell \neq 1$. Hence, $s(u_{ij} \otimes v_{k\ell}) = 0$ when $j \neq 1$ or $\ell \neq 1$, which completes the proof. $\square$

Using the embezzlement framework, we can distinguish the unitary correlation sets for qs and qc for all $n, m \ge 2$ and show that the unitary qs sets are not closed. The proof uses techniques found in [11, Theorem 2.1].

Corollary 3.3.4. *For every $n, m \ge 2$, $B_{qs}(n,m) \neq B_{qa}(n,m)$. In particular, $UC_{qs}(n,m) \neq UC_{qa}(n,m)$, and neither $UC_{qs}(n,m)$ nor $B_{qs}(n,m)$ are closed.*

Proof. Without loss of generality, we may assume that $n \le m$. Let $x = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i \in \mathbb{C}^n \otimes \mathbb{C}^m$. By Theorem 3.3.3, there is $X \in B_{qa}(n,m)$ with $X_{(i,1),(i,1)} = \frac{1}{\sqrt{n}}$ and $X_{(i,1),(j,1)} = 0$ for $i \neq j$. If $X \in B_{qs}(n,m)$, then there is a perfect embezzlement protocol in the tensor product model for $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$. Let $U_{ij}, V_{k\ell}$ and ψ be as in the perfect embezzlement framework. Then

$$(U \otimes I_m)(I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes \psi \otimes e_i.$$

Let $\alpha_1, \alpha_2, ...$ be the Schmidt coefficients of $e_1 \otimes \psi \otimes e_1$ with respect to the decomposition $(\mathbb{C}^n \otimes \mathcal{R}_A) \otimes (\mathcal{R}_B \otimes \mathbb{C}^m)$, so that

$$e_1 \otimes \psi \otimes e_1 = \sum_j \alpha_j x_j \otimes y_j,$$

where $\{x_j\} \subseteq \mathbb{C}^n \otimes \mathcal{R}_A$ and $\{y_j\} \subseteq \mathcal{R}_B \otimes \mathbb{C}^m$ are orthonormal sets. Since

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes \psi \otimes e_i = (U \otimes I_m)(I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \sum_j \alpha_j (U x_j) \otimes (V y_j),$$

the Schmidt coefficients of $e_1 \otimes \psi \otimes e_1$ must be the same as the Schmidt coefficients of $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes \psi \otimes e_i$. But if $\alpha_0 > 0$ is the largest Schmidt coefficient of $e_1 \otimes \psi \otimes e_1$, then the largest Schmidt coefficient of $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes \psi \otimes e_i$ is at most $\frac{1}{\sqrt{n}} \alpha_0$, which is a contradiction. Hence, $B_{qs}(n,m) \neq B_{qc}(n,m)$.

Finally, since any vector can be approximately embezzled, X must be a limit of elements in $B_{qs}(n,m)$, so that $B_{qs}(n,m)$ is not closed. It follows immediately that $UC_{qs}(n,m) \neq UC_{qa}(n,m)$ and that $UC_{qs}(n,m)$ is not closed. $\square$

Suppose that $n \le m$ and that $d_1, ..., d_n > 0$ are such that $\sum_{i=1}^n d_i^2 = 1$. Consider any state $s : \mathcal{V}_n \otimes_{\min} \mathcal{V}_m \to \mathbb{C}$ such that $s(u_{i1} \otimes v_{i1}) = d_i$ and $s(u_{i1} \otimes v_{k1}) = 0$ for $i \neq k$. Such a state arises from a perfect embezzlement protocol in the commuting model for the vector $\sum_{i=1}^n d_i e_i \otimes e_i$. A surprising fact about the state s is that its action on the elements $\{u_{ij} \otimes v_{k\ell}\}_{i,j,k,\ell}$ is necessarily unique.

Proposition 3.3.5. *Let $n \le m$ and let $d_1, ..., d_n > 0$ be such that $\sum_{i=1}^n d_i^2 = 1$. Suppose that $s : \mathcal{V}_n \otimes_c \mathcal{V}_m \to \mathbb{C}$ is a state such that $s(u_{i1} \otimes v_{i1}) = d_i$ for $1 \le i \le n$ and $s(u_{j1} \otimes v_{k1}) = 0$ for $j \neq k$. Then*

$$s(u_{ij} \otimes v_{k\ell}) = \begin{cases} d_i & i = k \le n,\ j = \ell = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let s be a state satisfying the equations given. In the embezzlement setting, s corresponds to the following: unitary operators $U : \mathbb{C}^n \otimes \mathcal{R} \to \mathbb{C}^n \otimes \mathcal{R}$ and $V : \mathcal{R} \otimes \mathbb{C}^m \to \mathcal{R} \otimes \mathbb{C}^m$ such that $(U \otimes I_m)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_m)$, along with a unit vector $\psi \in \mathcal{R}$ such that $s(u_{ij} \otimes v_{k\ell}) = \langle U_{ij} V_{k\ell} \psi, \psi \rangle$ for all i, j, k, ℓ. We may write the product of $U \otimes I_m$ and $I_n \otimes V$ in block form as

$$(U \otimes I_m)(I_n \otimes V) = (u_{ij} V)_{i,j=1}^n = (I_m \otimes V)(U \otimes I_n) = (v_{k\ell} U)_{k,\ell=1}^m.$$

With this identification in hand, one can check that

$$\langle U_{ij} V_{k\ell} \psi, \psi \rangle = (\langle (U \otimes I_m)(I_n \otimes V)(e_j \otimes \psi \otimes e_\ell), e_i \otimes \psi \otimes e_k \rangle.$$

By Proposition 3.3.1, we must have

$$U_{i1} V_{i1} \psi = d_i \psi,\ \forall 1 \le i \le n,$$

and similarly

$$U_{i1} V_{k1} \psi = 0,\ \forall i \neq k.$$

We aim to show that $\langle U_{ij} V_{k\ell} \psi, \psi \rangle = 0$ whenever $(i, j, k, \ell) \neq (i, 1, i, 1)$. We observe that

$$\begin{aligned} \sum_{j=1}^n \sum_{i=1}^n d_i e_i \otimes (U_{ij}^* \psi) \otimes e_j &= (U^* \otimes I_n) \left(\sum_{i=1}^n d_i e_i \otimes \psi \otimes e_i \right) \\ &= (I_n \otimes V)(e_1 \otimes \psi \otimes e_1) = \sum_{i=1}^n e_i \otimes (V_{i1} \psi) \otimes e_1. \end{aligned}$$

Comparing entries, we must have $U_{ij}^*\psi = 0$ for all $j \neq 1$. Similarly, if we instead apply $(I_n \otimes V^*)$, we obtain the following:

$$\sum_{k,\ell=1}^{m} d_\ell e_\ell \otimes (V_{k\ell}^*\psi) \otimes e_k = (I_n \otimes V^*) \left(\sum_{\ell=1}^{n} d_\ell e_\ell \otimes \psi \otimes e_\ell \right)$$
$$= (U \otimes I_n)(e_1 \otimes \psi \otimes e_1) = \sum_{k=1}^{n} e_1 \otimes U_{k1}\psi \otimes e_k.$$

Comparing entries shows that $V_{k\ell}^*\psi = 0$ if $\ell \neq 1$. At this point, it follows that if (i, j, k, ℓ) is not equal to $(i, 1, i, 1)$, then $\langle U_{ij}V_{k\ell}\psi, \psi\rangle = 0$, since $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ and one of $U_{ij}^*\psi = 0$ or $V_{k\ell}^*\psi = 0$. This completes the proof. □

This phenomenon applies to any maximally entangled unit vector in $\mathbb{C}^n \otimes \mathbb{C}^m$. Recall that any simple tensor $x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m$ has an associated map $T_{x,y} : \mathbb{C}^m \to \mathbb{C}^n$ given by $T_{x,y}(z) = \langle z, y\rangle x$. Extending by bilinearity, for any $\alpha \in \mathbb{C}^n \otimes \mathbb{C}^m$, there is an associated linear map $T_\alpha : \mathbb{C}^m \to \mathbb{C}^n$; moreover, this is a 1-1 correspondence. We will say that a unit vector $\alpha \in \mathbb{C}^n \otimes \mathbb{C}^m$ has **full Schmidt rank** if $\text{rank}(T_\alpha) = \min\{n, m\}$. Recall that any $x \in \mathbb{C}^n \otimes \mathbb{C}^m$ has a Schmidt decomposition $x = \sum_{i=1}^k d_i u_i \otimes v_i$, where $\{u_1, ..., u_k\} \subseteq \mathbb{C}^n$ is orthonormal and $\{v_1, ..., v_k\} \subseteq \mathbb{C}^m$ is orthonormal, and $d_i > 0$ are in decreasing order; moreover, the d_i are unique. Then a unit vector has full Schmidt rank if and only if $k = \min\{n, m\}$.

Corollary 3.3.6. *Let $\alpha \in \mathbb{C}^n \otimes \mathbb{C}^m$, and let $X \in B_{qc}(n, m)$ be any matrix obtained by a perfect embezzlement protocol for α in the commuting model. Then X is unique if and only if α has full Schmidt rank in $\mathbb{C}^n \otimes \mathbb{C}^m$.*

Proof. First, suppose that α has full Schmidt rank. Using the Schmidt decomposition, we write $\alpha = \sum_{i=1}^{\min(n,m)} d_i u_i \otimes v_i$, where $d_i > 0$ for all i. The proof of Proposition 3.3.5 shows that the embezzlement correlation is unique when $u_i = e_i$ and $v_i = e_i$. Thus, if $X \in B_{qc}(n, m)$ is a correlation matrix corresponding to a perfect embezzlement protocol for α, then we can apply a unitary of the form $A \otimes B \in M_n \otimes M_m$ that sends $u_i \otimes v_i$ to $e_i \otimes e_i$, and we obtain a correlation matrix corresponding to a perfect embezzlement protocol for $\sum_{i=1}^n d_i e_i \otimes e_i$. This correlation matrix is necessarily unique, so applying $A^* \otimes B^*$, the same result holds for α. Therefore, the matrix X is unique.

Conversely, suppose that $\text{rank}(T_\alpha) = p < \min\{n, m\}$. We may write the Schmidt decomposition $\alpha = \sum_{i=1}^p d_i u_i \otimes v_i$. Let $Y \in B_{qc}(p, p)$ be any matrix corresponding to a perfect embezzlement protocol for $\beta := \sum_{i=1}^p d_i e_i \otimes e_i$, and let $U = (U_{ij})$ and $V = (V_{k\ell})$ be unitaries in $M_n(\mathcal{B}(\mathcal{H}))$ and $M_m(\mathcal{B}(\mathcal{H}))$ respectively such that $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ for all i, j, k, ℓ. Now, the matrix $R = (U_{ij}) \oplus I_{n-p}$ is unitary in $M_n(\mathcal{B}(\mathcal{H}))$. Similarly, $S = (V_{k\ell}) \oplus I_{m-p}$ is unitary in $M_m(\mathcal{B}(\mathcal{H}))$, and the entries of R commute with the entries of S. Therefore, there is a state $s : \mathcal{U}_{nc}(n) \otimes_{\max} \mathcal{U}_{nc}(m) \to \mathbb{C}$ whose image in $B_{qc}(n,m)$ has top-left corner equal to Y and bottom-right entry equal to 1, and this will give rise to a perfect embezzlement protocol for β. Let $\widetilde{Y} \in B_{qc}(n,m)$ be the correlation corresponding to the state s. Now, let $A \in M_n$ and $B \in M_m$ be unitary matrices such that $Ae_i = u_i$ and $Be_i = v_i$ for $1 \leq i \leq p$. Using Proposition 3.1.4, $\| \cdot \|_{qc}$ is locally unitarily invariant. Thus, $X := (A \otimes B)\widetilde{Y} \in B_{qc}(n,m)$, and this corresponds to a perfect embezzlement protocol for α in the commuting model. If Z is the matrix obtained in Theorem 3.3.3 corresponding to β, then $(A \otimes B)Z$ is the matrix obtained in Theorem 3.3.3 corresponding to α. Since Z only has one non-zero column, we have $Z \neq \widetilde{Y}$. Since A and B are unitary, it follows that $(A \otimes B)\widetilde{Y} \neq (A \otimes B)Z$. Hence, there are two distinct correlations that give rise to perfect embezzlement protocols for α. It follows that the correlation matrix for α is not unique. $\square$

Corollary 3.3.7. *Let $\alpha = \sum \alpha_{i,k} e_i \otimes e_k \in \mathbb{C}^n \otimes \mathbb{C}^m$ be a state with full Schmidt rank, and let $X = (x_{(i,j),(k,l)}) \in M_n \otimes M_m$ with*

$$x_{(i,j),(k,l)} = \begin{cases} \alpha_{i,k}, & \text{when } j = l = 1 \\ 0, & \text{when } j \neq 1 \text{ or } l \neq 1. \end{cases}$$

Then X is an extreme point of $B_{qc}(n,m)$ and of $B_{qa}(n,m)$.

Proof. Since $X \in B_{qa}(n,m) \subseteq B_{qc}(n,m)$, we need only show that X is an extreme point of $B_{qc}(n,m)$. Suppose that $X = \frac{1}{2}(Y+Z)$ where $Y, Z \in B_{qc}(n,m)$. Let $\beta = \sum Y_{(i,1),(k,1)} e_i \otimes e_k$ and $\gamma = \sum Z_{(i,1),(k,1)} e_i \otimes e_k$. Then β and γ are vectors in $\mathbb{C}^n \otimes \mathbb{C}^m$ with norm at most 1. Moreover, $\alpha = \frac{1}{2}(\beta + \gamma)$. This forces $\beta = \gamma = \alpha$. In particular, Y and Z correspond to a perfect embezzlement protocol in the commuting model for α. Since α has full Schmidt rank, Corollary 3.3.6 shows that $Y = Z = X$. $\square$

We now give a characterization for elements of $B_{loc}(n,m)$ corresponding to perfect embezzlement protocols in the commuting model.

Theorem 3.3.8. *Let $\alpha = \sum_{i,j} \alpha_{ij} e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^m$ be a unit vector, where $n, m \geq 2$. The following are equivalent.*

1. *There is a state $s \in \mathcal{S}(\mathcal{V}_n \otimes_{\min} \mathcal{V}_m)$ such that $s(u_{i1} \otimes v_{j1}) = \alpha_{ij}$ for all i, j and $X := (s(u_{ij} \otimes v_{k\ell})) \in B_{loc}(n, m)$.*

2. *There exist unit $t_1, ..., t_s \geq 0$ such that $\sum_{r=1}^s t_r = 1$, and unit vectors $y_1, ..., y_s \in \mathbb{C}^n$ and $z_1, ..., z_s \in \mathbb{C}^m$ such that*
$$\alpha = \sum_{r=1}^s t_r y_r \otimes z_r.$$

3. $\|\alpha\|_{\mathbb{C}^n \otimes_\pi \mathbb{C}^m} = 1$.

Proof. Since α is norm 1 in the Hilbert space tensor product $\mathbb{C}^n \otimes \mathbb{C}^m$, we must have $\|\alpha\|_\pi \geq 1$. Clearly by definition of the projective tensor product, (2) implies (3). Suppose that (3) is true. By Proposition 1.2.2, the open ball of radius $R > 0$ about 0 in $\mathbb{C}^n \otimes_\pi \mathbb{C}^m$ is the convex hull of the set $\{x \otimes y \in \mathbb{C}^n \otimes \mathbb{C}^m : \|x\|\|y\| < R\}$. For each $R > 1$, we may write $\alpha = \sum_{r=1}^s t_r y_r \otimes z_r$ for some $t_1, ..., t_s \geq 0$ with $\sum_{r=1}^s t_r = 1$ and vectors $y_1, ..., y_s \in \mathbb{C}^n$ and $z_1, ..., z_s \in \mathbb{C}^m$ such that $\|y_r\|\|z_r\| \leq R$ for all r. By a theorem of Caratheodory, we may always assume that $s \leq 2\dim(\mathbb{C}^n \otimes \mathbb{C}^m) + 1 = 2nm + 1$. Since each $t_r \leq 1$ and $\|y_r\|, \|z_r\| < R$, by compactness and letting $R \to 1$, we may write
$$\alpha = \sum_{r=1}^s t_r y_r \otimes z_r,$$
where $t_1, ..., t_s \geq 0$ with $\sum_{r=1}^s t_r = 1$ and $\|y_r\| = \|z_r\| = 1$. Therefore, $\|\alpha\|_\pi \leq 1$, so that $\|\alpha\|_\pi = 1$. It follows that (2) and (3) are equivalent.

Suppose that (1) holds, and let $X \in B_{loc}(n, m)$ be such that $X_{(i,1),(j,1)} = \alpha_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We assume without loss of generality that $n \leq m$. Since the first column of X is of norm 1, we have $\|X\| \geq 1$; in particular, $\|X\|_\pi \geq 1$. Therefore, $\|X\|_\pi = 1$. Let $P_n : M_n \to \mathbb{C}^n$ and $P_m : M_m \to \mathbb{C}^m$ be the linear maps defined by sending a matrix to its first column. Then P_n and P_m are contractive. Since the projective Banach space tensor norm is functorial, $P_n \otimes P_m : M_n \otimes_\pi M_m \to \mathbb{C}^n \otimes_\pi \mathbb{C}^m$ is contractive. We observe that $(P_n \otimes P_m)(X) = \alpha$, so that $\|\alpha\|_{\mathbb{C}^n \otimes_\pi \mathbb{C}^m} \leq 1$. The reverse inequality is immediate since $\|\alpha\|_{\mathbb{C}^{nm}} = 1$, which shows that (1) implies (3).

Suppose that (3) is true. Let $X \in B_{qa}(n, m)$ be the matrix obtained in Theorem 3.3.3 corresponding to a perfect embezzlement protocol in the commuting model for α. Then X is a matrix with 0's in every column except that the first column has the entries of α. The inclusion maps $\iota_n : \mathbb{C}^n \to M_n$ and $\iota_m : \mathbb{C}^m \to M_m$ obtained by sending a vector x to the matrix of 0's with first column x are contractive, so $\iota_n \otimes \iota_m : \mathbb{C}^n \otimes_\pi \mathbb{C}^m \to M_n \otimes_\pi M_m$

is contractive. Moreover, $(\iota_n \otimes \iota_m)(\alpha) = X$, which forces $X \in B_{loc}(n,m)$. This shows that (3) implies (1). $\square$

Since the state corresponding to perfect embezzlement in the commuting model for $\frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i$ is unique and takes the form given in Theorem 3.3.3, we can separate $B_{loc}(n,m)$ and $B_q(n,m)$; moreover, we can also separate $B_{qc}(n,m)$ and $B_{qmax}(n,m)$.

Corollary 3.3.9. *For all $n, m \geq 2$, we have $UC_{loc}(n,m) \subsetneq UC_q(n,m)$.*

Proof. As usual, we may assume that $n \leq m$. Let X be the matrix obtained from the state $s \in \mathcal{S}(\mathcal{V}_n \otimes_{\min} \mathcal{V}_m)$ in Proposition 3.3.5. By Theorem 3.3.8, $X \in B_{loc}(n,m)$ if and only if $\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^n e_i \otimes e_i \right\|_{\mathbb{C}^n \otimes_\pi \mathbb{C}^m} = 1$. To see that this is not the case, Let $B : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}$ be the bilinear form given by

$$B(v,w) = \sum_{k=1}^n v_k w_k.$$

By Holder's inequality, $\|B\| \leq 1$ when regarded as a bilinear form from $\mathbb{C}^n \times \mathbb{C}^m$ into $\mathbb{C}$. It follows by Theorem 1.2.3 that $\|X\|_\pi \geq \left| \sum_{i=1}^n \frac{1}{\sqrt{n}} B(e_i, e_i) \right| = \sqrt{n}$. Hence, $X \notin B_{loc}(n,m)$, which shows that $UC_{loc}(n,m) \neq UC_{qa}(n,m)$. Since X can be approximated by elements in $B_q(n,m)$ and $B_{loc}(n,m)$ is closed, we see that $B_{loc}(n,m) \neq B_q(n,m)$, so that $UC_{loc}(n,m) \neq UC_q(n,m)$.

Finally, we will show that $UC_{loc}(n,m) \subset UC_q(n,m)$. We first note that $UC_{loc}(n,m) \subseteq \mathbb{C}^{(2n^2+1)(2m^2+1)}$ is the closed convex hull of states arising from evaluation functionals on commutative C^*-algebras. As in the proof of Theorem 3.1.3, the resulting correlation in $UC_{loc}(n,m)$ will be of the form

$$(\delta_z(X)\delta_z(Y))_{X \in \mathfrak{B}_n(U),\, Y \in \mathfrak{B}_m(V)},$$

where $U \in M_n(\mathcal{B}(\mathcal{H}))$ and $V \in M_m(\mathcal{B}(\mathcal{H}))$ are unitary and K is a compact Hausdorff space such that $z \in K$ and $C^*(\mathfrak{B}_n(U) \cup \mathfrak{B}_m(V)) \simeq C(K)$. We saw in the proof of Theorem 3.1.3 that $(\delta_z(X)\delta_z(Y))_{X,Y} \in UC_q(n,m)$. By a theorem of Caratheodory, every element of $UC_{loc}(n,m)$ can be written as a finite convex combination of at most $2(2n^2+1)(2m^2+1)+1$ states of the form $(\delta_z(X)\delta_z(Y))_{X,Y}$. Since $UC_q(n,m)$ is convex, it follows that $UC_{loc}(n,m) \subseteq UC_q(n,m)$, which completes the proof. $\square$

Corollary 3.3.10. *For all $n, m \geq 2$, $B_{qc}(n,m) \neq B_{qmax}(n,m)$. In particular, $\mathcal{V}_n \otimes_c \mathcal{V}_m \neq \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$. In fact, the identity map $id : \mathcal{V}_n \otimes_c \mathcal{V}_m \to \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$ fails to be 1-positive.*

Proof. The extreme points of $B_{qmax}(n,m)$ are the extreme points of the unit ball of $M_n \otimes M_m$ in the operator norm, which is just the set of unitaries in M_{nm}. By Corollary 3.3.7, there are proper contractions in $B_{qc}(n,m)$ that are extreme in $B_{qc}(n,m)$. Therefore, $B_{qc}(n,m) \neq B_{qmax}(n,m)$. This shows that $\mathrm{id} : \mathcal{V}_n \otimes_c \mathcal{V}_m \to \mathcal{V}_n \otimes_{\max} \mathcal{V}_m$ fails to be 1-positive. $\square$

Corollary 3.3.11. *None of the norms $\|\cdot\|_{loc}$, $\|\cdot\|_{qa}$ or $\|\cdot\|_{qc}$ are unitarily invariant.*

Proof. There is a unitary $W \in M_n \otimes M_m$ with $W \notin B_{qc}(n,m)$; otherwise, we would have $B_{qmax}(n,m) \subseteq B_{qc}(n,m)$, since $B_{qmax}(n,m)$ is the closed convex hull of the unitaries in $M_n \otimes M_m$. Note that $I_{nm} = I_n \otimes I_m \in B_{loc}(n,m)$ by Theorem 3.1.5. However, $I_{nm}W = W \notin B_{qc}(n,m)$, so that $\|I_{nm}W\|_t > 1$ for $t \in \{loc, qa, qc\}$. Hence, $\|\cdot\|_{loc}$, $\|\cdot\|_{qa}$ and $\|\cdot\|_{qc}$ are not unitarily invariant. $\square$

Chapter 4

Connes' embedding problem and winning strategies for quantum XOR games

In this chapter, we show that the class of quantum XOR games (defined in [54]) is rich enough to detect the solution to Connes' embedding problem. In particular, determining whether every quantum XOR game has the same optimal winning probability in the commmuting model as in the approximate finite-dimensional model is equivalent to Connes' embedding problem. In fact, the unitary correlation sets defined in Chapter 3 encode the possible strategies for quantum XOR games. One way to deduce that $B_{qs}(n, m)$ is not closed for all $n, m \geq 2$ is by using the coherent embezzlement game from [54]. In light of these facts, it is plausible that studying quantum XOR games may be a reasonable plan of attack for solving the embedding problem.

First, we will give a brief overview of quantum XOR games from [54] and the notion of bias for these games. We also show the correspondence between bias and linear functionals on $M_n \otimes M_n$ that are contractive with respect to the unitary correlation norms from Chapter 3. We will then use Lemma 4.2.1 to reduce the Tsirelson problem for unitary correlations to self-adjoint unitary correlations. This allows us to prove Corollary 4.2.3, which states the equivalence of the embedding problem to the problem of optimal strategies for quantum XOR games in the commuting and approximate finite-dimensional models.

4.1 Introduction to quantum XOR games

We will briefly introduce the definition of a quantum XOR game with two parties, Alice and Bob. More information on this class of extended non-local games can be found in [54]. Loosely speaking, a quantum XOR game is a generalization of a classical XOR game. In the classical case, the referee has a list of n possible questions $\{1, ..., n\}$, and the set of possible answers for Alice and Bob is $\{0, 1\}$. For each pair $s, t \in \{1, ..., n\}$, there is some associated number $R_{s,t} \in [-1, 1]$ (known to Alice and Bob) satisfying $\sum_{s,t} |R_{s,t}| = 1$. The referee gives question s to Alice and question t to Bob with probability $|R_{s,t}|$. If $R_{s,t} \geq 0$, then Alice and Bob must respond with the same bit; if $R_{s,t} < 0$, then they must respond with different bits.

In a quantum XOR game, the questions are now given as states (i.e., unit vectors) on a certain Hilbert space. In particular, the referee sends some state on $\mathbb{C}^n \otimes \mathbb{C}^n$ to Alice and Bob, where Alice has access to the left copy of $\mathbb{C}^n$ and Bob has access to the right copy of $\mathbb{C}^n$. Every quantum XOR game of size n is associated with a self-adjoint matrix $M \in M_n \otimes M_n$ with $\|M\|_1 \leq 1$, where $\| \cdot \|_1$ denotes the trace norm. Conversely, every self-adjoint matrix $M \in M_n \otimes M_n$ with $\|M\|_1 \leq 1$ is associated to a quantum XOR game G of size n [54]. For the sake of simplicity, we will always consider the case where $\|M\|_1 = 1$.

For our purposes, a quantum XOR game G of size n can be described as follows (see [54] for a more general definition): let $\{\varphi_i\}_{i=1}^{n^2} \subseteq \mathbb{C}^n \otimes \mathbb{C}^n$ be an orthonormal basis. Let $p_1, ..., p_{n^2} \in [0, 1]$ be such that $\sum_{i=1}^{n^2} p_i = 1$, and let $c_i \in \{0, 1\}$ for each $1 \leq i \leq n^2$. With probability p_i, the referee prepares the state $\varphi_i \in \mathbb{C}^n \otimes \mathbb{C}^n$. Alice and Bob may use their own Hilbert space $\mathcal{H}$ and **observables** $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ (i.e., self-adjoint unitaries) such that $U \otimes I_n$ and $I_n \otimes V$ commute in $\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H} \otimes \mathbb{C}^n)$. They may also prepare their space in the state $\psi \in \mathcal{H}$. Based on the application of U and V to the state ψ, Alice and Bob return outcomes $a \in \{0, 1\}$ and $b \in \{0, 1\}$ respectively. If $c_i = 0$, then Alice and Bob's output bits must be equal; if $c_i = 1$, their output bits must be distinct. (If Alice and Bob are working in the tensor product model, then there must be a decomposition $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ where $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}_A)$ and $V \in \mathcal{B}(\mathcal{H}_B \otimes \mathbb{C}^n)$, and where $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is a state. Moreover, the operator $(U \otimes I_n)(I_n \otimes V)$ is replaced with $U \otimes V$.)

For a quantum XOR game G as above, the matrix associated with G is given by $M = \sum_{i=1}^{n^2} (-1)^{c_i} p_i \varphi_i \varphi_i^*$, where $\varphi_i \varphi_i^*$ denotes the rank one orthogonal projection of $\mathbb{C}^n \otimes \mathbb{C}^n$ onto span $\{\varphi_i\}$. The following result allows us to translate between quantum XOR games and certain self-adjoint matrices.

Proposition 4.1.1. (Regev-Vidick, [54]) *Let G be a quantum XOR game of size n. Then the matrix $M \in M_n \otimes M_n$ associated with G is self-adjoint and $\|M\|_1 = 1$. Conversely, if*

$M \in M_n \otimes M_n$ is self-adjoint with $\|M\|_1 = 1$, then there is a quantum XOR game G with associated matrix M.

We recall that, by Theorem 3.1.5, for $t \in \{qa, qc\}$, the set $B_t(n,n)$ is the unit ball of a reasonable cross-norm on $M_n \otimes M_n$. We will denote this norm by $\|\cdot\|_t$, and we will denote the Banach space $(M_n \otimes M_n, \|\cdot\|_t)$ by $M_n \otimes_t M_n$. Finally, we will let $\|\cdot\|_t^*$ denote the dual norm of $\|\cdot\|_t$ on $(M_n \otimes M_n)^*$.

For a quantum XOR game G and $t \in \{q, qa, qc\}$, we define a ***t*-strategy** for Alice and Bob to be a correlation $X \in B_t(n,n)$.

Instead of working with maximum success probabilities in different models, it is convenient to work with a related quantity, known as the bias. To ease notation, whenever $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and $\mathcal{H}$ is finite-dimensional, we will denote by $\mathrm{Tr}_{\mathcal{H}}$ the operator $\mathrm{Tr} \otimes \mathrm{id}_{\mathcal{K}}$ acting on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$, where Tr denotes the unnormalized trace on $\mathcal{B}(\mathcal{H})$. With this in hand, the **entanglement bias** (or the **quantum bias**) of a quantum XOR game G with associated matrix M is defined by

$$\omega_q^*(G) = \sup\{\langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[(U \otimes V)(M \otimes I_{\mathcal{H}_A \otimes \mathcal{H}_B})]\psi, \psi\rangle\},$$

where the supremum is taken over all finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, unit vectors $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$, and observables $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}_A)$ and $V \in \mathcal{B}(\mathcal{H}_B \otimes \mathbb{C}^n)$ (that is, self-adjoint unitaries). In the supremum above, we are identifying $M \otimes I_{\mathcal{H}_A \otimes \mathcal{H}_B}$ with the matrix $M' \in M_{n^2}(\mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B))$ given by $M' = (M_{(i,j),(k,\ell)} I_{\mathcal{H}_A \otimes \mathcal{H}_B})_{(i,j),(k,\ell)}$.

Similarly, we may define the **commuting bias** of a quantum XOR game G with associated matrix M to be

$$\omega_{qc}^*(G) = \sup\{\langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[((U \otimes I_{\mathbb{C}^n})(I_{\mathbb{C}^n} \otimes V)(M \otimes I_{\mathcal{H}})]\psi, \psi\rangle,$$

where the supremum is taken over all Hilbert spaces $\mathcal{H}$, unit vectors $\psi \in \mathcal{H}$, and self-adjoint unitaries $U = (U_{ij}) \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V = (V_{k\ell}) \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ such that $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$ as operators on $\mathbb{C}^n \otimes \mathcal{H} \otimes \mathbb{C}^n$. Here, we are identifying $M \otimes I_{\mathcal{H}}$ with the operator $M' \in M_{n^2}(\mathcal{B}(\mathcal{H}))$ given by $M' = (M_{(i,j),(k,\ell)} I_{\mathcal{H}})$. Adapting the proof of Proposition 3.3.1, since the matrix $(U \otimes I_n)(I_n \otimes V)$ is given by $(U_{ij} V_{k\ell})_{(i,j),(k,\ell)}$, it follows that $U \otimes I_n$ commutes with $I_n \otimes V$ if and only if $U_{ij} V_{k\ell} = V_{k\ell} U_{ij}$ for all i, j, k, ℓ. In particular, since $U = U^*$ and $V = V^*$, we have $U_{ij} = U_{ji}^*$ and $V_{k\ell} = V_{\ell k}^*$. Thus, the self-adjoint unitaries U and V satisfy $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$ if and only if the set $\{I\} \cup \{U_{ij}\}_{i,j=1}^n$ $*$-commutes with the set $\{I\} \cup \{V_{k\ell}\}_{k,\ell=1}^n$.

One may view both of these notions of bias as twice the difference between the maximum success probability in the corresponding model and the success probability from the random strategy (i.e. Alice and Bob respond randomly, regardless of the input). Although the argument appears in [54], we reproduce it here for convenience. Suppose that Alice and Bob use the qc-strategy (U, V, ψ), where $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ are observables, and $\psi \in \mathcal{H}$ is a unit vector. For simplicity, we first consider the case when the referee prepares the state $\varphi \in \mathbb{C}^n \otimes \mathbb{C}^n$, with associated number $c \in \{0,1\}$. We write $U = P_0 - P_1$ and $V = Q_0 - Q_1$, where $P_0, P_1 \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $Q_0, Q_1 \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ are orthogonal projections with $P_0 + P_1 = I_{\mathbb{C}^n \otimes \mathcal{H}}$ and $Q_0 + Q_1 = I_{\mathcal{H} \otimes \mathbb{C}^n}$. The projections P_0, Q_0 correspond to the output 0 for Alice and Bob, respectively; while the projections P_1, Q_1 correspond to the output 1 for Alice and Bob, respectively. Alice and Bob's observables are applied to the state $\psi \otimes \varphi$. If $c = 0$, then Alice and Bob's outputs must be equal. Hence, the probability of Alice and Bob winning the game is

$$\begin{aligned}
p(a = b|\varphi) &= p(0,0|\varphi) + p(1,1|\varphi) \\
&= \langle (P_0 \otimes I_n)(I_n \otimes Q_0)(\psi \otimes \varphi), \psi \otimes \varphi \rangle + \langle (P_1 \otimes I_n)(I_n \otimes Q_1)(\psi \otimes \varphi), \psi \otimes \varphi \rangle \\
&= \frac{1}{2}(\langle (I_n \otimes I_{\mathcal{H}} \otimes I_n)(\psi \otimes \varphi), \psi \otimes \varphi \rangle + \langle (U \otimes I_n)(I_n \otimes V)(\psi \otimes \varphi), \psi \otimes \varphi \rangle) \\
&= \frac{1}{2}(1 + \langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[(U \otimes I_n)(I_n \otimes V)(\varphi\varphi^* \otimes I_{\mathcal{H}})]\psi, \psi \rangle.)
\end{aligned}$$

Similarly, if $c = 1$, then Alice and Bob must respond with distinct bits. In this case, the probability of Alice and Bob winning the game is

$$\begin{aligned}
p(a \neq b|\varphi) &= p(0,1|\varphi) + p(1,0|\varphi) \\
&= \langle (P_0 \otimes I_n)(I_n \otimes Q_1)(\psi \otimes \varphi), \psi \otimes \varphi \rangle + \langle (P_1 \otimes I_n)(I_n \otimes Q_0)(\psi \otimes \varphi), \psi \otimes \varphi \rangle \\
&= \frac{1}{2}(\langle (I_n \otimes I_{\mathcal{H}} \otimes I_n)(\psi \otimes \varphi), \psi \otimes \varphi \rangle - \langle (U \otimes I_n)(I_n \otimes V)(\psi \otimes \varphi), \psi \otimes \varphi \rangle) \\
&= \frac{1}{2}(1 + \langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[(U \otimes I_n)(I_n \otimes V)(-\varphi\varphi^* \otimes I_{\mathcal{H}})]\psi, \psi \rangle.)
\end{aligned}$$

Therefore, the probability of winning the game, given that φ was the input state, is

$$\frac{1}{2}(1 + \langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[(U \otimes I_n)(I_n \otimes V)((-1)^c\varphi\varphi^* \otimes I_{\mathcal{H}})]\psi, \psi \rangle).$$

In the general setup of a quantum XOR game, the referee has an orthonormal basis $\{\varphi_i\}_{i=1}^{n^2}$ for $\mathbb{C}^n \otimes \mathbb{C}^n$ with associated probability density $(p_i)_{i=1}^{n^2}$ and numbers $\{c_i\}_{i=1}^{n^2}$ with each $c_i \in \{0,1\}$. It is not hard, then, to see that the probability of winning with the strategy (U, V, ψ) is

$$\frac{1}{2}(1 + \langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n}[(U \otimes I_n)(I_n \otimes V)(M \otimes I_{\mathcal{H}})]\psi, \psi \rangle),$$

where $M = \sum_{i=1}^{n^2} (-1)^{c_i} p_i \varphi_i \varphi_i^*$. Since the random strategy has success probability $\frac{1}{2}$, it is easy to see that, if p is the probability that Alice and Bob win with the strategy (U, V, ψ), then the bias for this strategy is $2p - 1$. Thus, if p is the maximum success probability of winning a quantum XOR game G using t-strategies (for $t \in \{q, qc\}$), then $\omega_t^*(G) = 2p - 1$.

We will also consider the above definitions of bias that arise from omitting the assumption that U and V are self-adjoint, while keeping the other assumptions intact. In particular, we may consider the notions of bias given with respect to the unitary correlation sets defined above. In this context, we also consider the bias of a particular strategy, which has the same definition but is denoted by $\omega_t^*(G, U, V, \psi)$ for a specific strategy (U, V, ψ) whose correlation matrix X is in $B_t(n, n)$. If X is the correlation associated with (U, V, ψ), then we also let $\omega_t^*(G, X) = \omega_t^*(G, U, V, \psi)$. We note that $\omega_t^*(G, U, V, \psi)$ is $\mathbb{C}$-valued. A **perfect t-strategy** is a t-strategy X for which $\omega_t^*(G, X) = 1$. It will follow from Theorem 4.1.5 that there is a perfect t-strategy for the quantum XOR game G if and only if there is a t-strategy arising from observables for which the probability that Alice and Bob win is 1.

We remark that there is a natural correspondence between bias for quantum XOR games and self-adjoint linear functionals on $M_n \otimes M_n$. The easiest way to see this is using unitary correlation sets. Indeed, suppose that $X = (\langle U_{ij} V_{k\ell} \psi, \psi \rangle)_{(i,j),(k,\ell)} \in B_{qc}(n, n)$. Then since M is self-adjoint,

$$\begin{aligned}
\omega_{qc}^*(G, X) &= \langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n} [(U_{ij} V_{k\ell})_{(i,j),(k,\ell)} (M_{(i,j),(k,\ell)} I_{\mathcal{H}})_{(i,j),(k,\ell)}] \psi, \psi \rangle \\
&= \left\langle \mathrm{Tr}_{\mathbb{C}^n \otimes \mathbb{C}^n} \left(\sum_{p,q=1}^{n} U_{ip} V_{kq} M_{(p,j),(q,\ell)} \right) \psi, \psi \right\rangle \\
&= \sum_{i,j,p,q} \langle U_{ip} V_{jq} M_{(p,i),(q,j)} \psi, \psi \rangle \\
&= \sum_{i,j,p,q} M_{(p,i),(q,j)} \langle U_{ip} V_{jq} \psi, \psi \rangle \\
&= \sum_{i,j,k,\ell} M_{(j,i),(\ell,k)} X_{(i,j),(k,\ell)}, \\
&= \mathrm{Tr}(MX).
\end{aligned}$$

Thus, the quantum XOR game G defines the linear functional $\mathrm{Tr}(M \cdot) : M_n \otimes M_n \to \mathbb{C}$ which gives $\omega_{qc}^*(G, X)$ for each $X \in B_{qc}(n, n)$. An analogous argument holds for $\omega_q^*(G, X)$ whenever the corresponding correlation matrix X lies in $B_q(n, n)$.

A helpful fact is that for $t \in \{q, qs, qa, qc\}$, the self-adjoint t-correlations arise from t-strategies involving observables.

Proposition 4.1.2. *Let $t \in \{q, qs, qa, qc\}$ and $X = X^* \in B_t(n,n)$. If $t = qc$, then there is a Hilbert space $\mathcal{H}$, self-adjoint unitaries $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ and a unit vector $\psi \in \mathcal{H}$ such that $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$ and $X_{(i,j),(k,\ell)} = \langle U_{ij} V_{k\ell} \psi, \psi \rangle$ for all i,j,k,ℓ. If $t = qs$, then we may take $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and self-adjoint unitaries $U \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}_A)$ and $V \in \mathcal{B}(\mathcal{H}_B \otimes \mathbb{C}^n)$ such that $X_{(i,j),(k,\ell)} = \langle (U_{ij} \otimes V_{k\ell})\psi, \psi \rangle$ for all i,j,k,ℓ. Moreover, if $t = q$, then we may take $\mathcal{H}_A$ and $\mathcal{H}_B$ to be finite-dimensional. Finally, if $t = qa$, then $X = \lim_{m \to \infty} Y^{(m)}$, where $Y^{(m)} \in B_q(n,n)$ is a q-strategy involving observables.*

Proof. We first let $t = q$, so that $X = (\langle (R_{ij} \otimes S_{k\ell})\psi, \psi \rangle)_{(i,j),(k,\ell)}$ for unitaries $R = (R_{ij}) \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H}_A)$, $S = (S_{k\ell}) \in \mathcal{B}(\mathcal{H}_B \otimes \mathbb{C}^n)$, finite-dimensional Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, and a unit vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let $U_{ij} = \begin{bmatrix} 0 & R_{ij} \\ R_{ji}^* & 0 \end{bmatrix}$ and $V_{k\ell} = \begin{bmatrix} 0 & S_{k\ell} \\ S_{\ell k}^* & 0 \end{bmatrix}$. Performing a canonical shuffle on the unitaries $\begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix} \in M_2(M_n(\mathcal{B}(\mathcal{H}_A)))$ and $\begin{bmatrix} 0 & S \\ S^* & 0 \end{bmatrix} \in M_2(M_n(\mathcal{B}(\mathcal{H}_B)))$, we see that $U = (U_{ij})$ and $V = (V_{k\ell})$ are self-adjoint unitaries in $M_n(\mathcal{B}(\mathcal{H}_A \oplus \mathcal{H}_A))$ and $M_n(\mathcal{B}(\mathcal{H}_B \oplus \mathcal{H}_B))$, respectively. Hence, we obtain a q-strategy $(U, V, \widetilde{\psi})$, where $\widetilde{\psi} = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi & 0 & 0 & \psi \end{bmatrix}^t$. Using the fact that $U_{ij}^* = U_{ji}$ and $V_{k\ell}^* = V_{\ell k}$,

$$\langle (U_{ij} \otimes V_{k\ell})\widetilde{\psi}, \widetilde{\psi} \rangle = \frac{1}{2} \langle (R_{ij} \otimes S_{k\ell})\psi, \psi \rangle + \frac{1}{2} \langle (R_{ji}^* \otimes S_{\ell k}^*)\psi, \psi \rangle = X_{(i,j),(k,\ell)}.$$

The proof for $t = qs$ is similar. For $t = qc$, we assume that $R \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ are unitaries and $\psi \in \mathcal{H}$ is a unit vector such that $(R \otimes I_n)(I_n \otimes S) = (I_n \otimes S)(R \otimes I_n)$ and $X_{(i,j),(k,\ell)} = \langle R_{ij} S_{k\ell} \psi, \psi \rangle$ for all i,j,k,ℓ. We let

$$U_{ij} = \begin{bmatrix} 0 & R_{ij} & 0 & 0 \\ R_{ji}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & R_{ij} \\ 0 & 0 & R_{ji}^* & 0 \end{bmatrix} \text{ and } V_{k\ell} = \begin{bmatrix} 0 & 0 & S_{k\ell} & 0 \\ 0 & 0 & 0 & S_{k\ell} \\ S_{\ell k}^* & 0 & 0 & 0 \\ 0 & S_{\ell k}^* & 0 & 0 \end{bmatrix}.$$

A calculation shows that

$$U_{ij} V_{k\ell} = \begin{bmatrix} 0 & 0 & 0 & R_{ij} S_{k\ell} \\ 0 & 0 & R_{ji}^* S_{k\ell} & 0 \\ 0 & R_{ij} S_{\ell k}^* & 0 & 0 \\ R_{ji}^* S_{\ell k}^* & 0 & 0 & 0 \end{bmatrix} = V_{k\ell} U_{ij},$$

so that $U = (U_{ij}) \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V = (V_{k\ell}) \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ are self-adjoint unitaries with $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$. Letting $\widetilde{\psi} = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi & 0 & 0 & \psi \end{bmatrix}^t$, it readily follows that

$$\langle U_{ij} V_{k\ell} \psi, \psi \rangle = \frac{1}{2} \langle R_{ij} S_{k\ell} \psi, \psi \rangle + \frac{1}{2} \langle R_{ji}^* S_{\ell k}^* \psi, \psi \rangle = X_{(i,j),(k,\ell)}.$$

Thus, the proposition holds for $t = qc$. The last statement about qa correlations immediately follows from the $t = q$ case. $\square$

The next proposition, combined with convexity of each $B_t(n,n)$, guarantees that the real part of X is in $B_t(n,n)$ whenever $X \in B_t(n,n)$.

Proposition 4.1.3. *Let $n \geq 2$ and $t \in \{q, qs, qa, qc\}$. If $X = (X_{(i,j),(k,\ell)})$ belongs to $B_t(n,n)$, then $X^* \in B_t(n,n)$.*

Proof. Suppose that $U = (U_{ij}) \in M_n(\mathcal{B}(\mathcal{H}))$ and $V = (V_{k\ell}) \in M_n(\mathcal{B}(\mathcal{H}))$ are unitary and $\psi \in \mathcal{H}$ is a unit vector such that $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ for all i, j, k, ℓ and

$$\langle U_{ij}V_{k\ell}\psi, \psi\rangle = X_{(i,j),(k,\ell)}.$$

Then

$$X^* = (\overline{X}_{(j,i),(\ell,k)}) = (\overline{\langle U_{ji}V_{\ell k}\psi, \psi\rangle}) = (\langle \psi, U_{ji}V_{\ell k}\psi\rangle) = (\langle U_{ji}^* V_{\ell k}^* \psi, \psi\rangle),$$

using the fact that $U_{ij}^* V_{k\ell}^* = V_{k\ell}^* U_{ij}^*$ for all i, j, k, ℓ. It follows that

$$X^* = (\langle (U^*)_{ij}(V^*)_{k\ell}\psi, \psi\rangle)_{(i,j),(k,\ell)} \in B_{qc}(n,n).$$

A similar argument gives the desired result when $t \in \{q, qs\}$. The same result follows for $t = qa$ by using the density of $B_q(n,n)$ in $B_{qa}(n,n)$. $\square$

Corollary 4.1.4. *For $n \geq 2$ and $t \in \{loc, q, qs, qa, qc\}$, let $\| \cdot \|_t^*$ denote the dual norm on $M_n \otimes M_n$ with respect to the normed space $M_n \otimes_t M_n$. Then $\| \cdot \|_t$ is a $*$-norm; i.e., if $\|M\|_t^* \leq 1$, then $\|M^*\|_t^* \leq 1$.*

Proof. Let $M \in M_n \otimes M_n$ be such that $\|M\|_t^* \leq 1$. The linear functional on $M_n \otimes M_n$ with respect to M is given by

$$f((X_{(i,j),(k,\ell)})) = \mathrm{Tr}(XM).$$

If g is the linear functional on $M_n \otimes M_n$ with respect to M^*, then

$$g((X_{(i,j),(k,\ell)})) = \mathrm{Tr}(XM^*) = \overline{\mathrm{Tr}(MX^*)} = \overline{\mathrm{Tr}(X^*M)} = \overline{f(X^*)}.$$

Since $\| \cdot \|_t$ is a $*$-norm, it follows that $\|M^*\|_t^* \leq 1$, as required. $\square$

Using Propositions 4.1.2 and 4.1.3, we obtain an equivalent description of bias, which allows us to use the theory of unitary correlations.

Theorem 4.1.5. *Let G be a quantum XOR game of size n with associated matrix $M \in M_n \otimes M_n$. Then*

$$\omega_{qc}^*(G) = \sup\{|\mathit{Tr}(MX)| : X \in B_{qc}(n,n)\}.$$

Similarly, we have

$$\omega_q^*(G) = \sup\{|\mathit{Tr}(MX)| : X \in B_q(n,n)\}.$$

Proof. Since every observable is a self-adjoint unitary, it is clear that $\omega_{qc}^*(G)$ is at most the quantity given in the theorem statement; thus, we need only establish the reverse inequality. Using the fact that $B_{qc}(n,n)$ is compact, we may choose $X \in B_{qc}(n,n)$ such that

$$\sup\{|\mathrm{Tr}(MZ)| : Z \in B_{qc}(n,n)\} = |\mathrm{Tr}(MX)|.$$

Suppose that $X = \langle U_{ij}V_{k\ell}\psi, \psi\rangle$ where $U = (U_{ij}) \in \mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$ and $V = (V_{k\ell}) \in \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ are unitaries, $\psi \in \mathcal{H}$ is a unit vector and $(U \otimes I_n)(I_n \otimes V) = (I_n \otimes V)(U \otimes I_n)$. By multiplying the unitary U by some $\lambda \in \mathbb{T}$ if necessary, we may assume that

$$\mathrm{Tr}(MX) = \omega_{qc}^*(G,X) = \sup\{|\mathrm{Tr}(MZ)| : Z \in B_{qc}(n,n)\}.$$

Since $B_{qc}(n,n)$ is convex, it follows that $Y := \frac{1}{2}(X + X^*) \in B_{qc}(n,n)$. By Proposition 4.1.2, Y is represented by self-adjoint observables. Finally, we see that

$$\begin{aligned}
\omega_{qc}^*(G,Y) &= \mathrm{Tr}\left(\frac{1}{2}(X+X^*)M\right) \\
&= \frac{1}{2}(\mathrm{Tr}(XM) + \mathrm{Tr}(X^*M)) \\
&= \frac{1}{2}(\mathrm{Tr}(XM) + \overline{\mathrm{Tr}(XM)}) \\
&= \omega_{qc}^*(G),
\end{aligned}$$

using the fact that M is self-adjoint. This establishes the reverse inequality for the commuting case. For the entanglement bias, the same argument shows that if $X \in B_q(n,n)$ with $\omega_q(G,X) = \alpha \in [0,1]$, then $Y := \frac{1}{2}(X + X^*) \in B_q(n,n)$ satisfies $\omega_q(G,Y) = \alpha$. Using Proposition 4.1.2 and taking the supremum over all such strategies, the result holds for the entanglement bias. $\square$

4.2 Relating quantum XOR games to Connes' embedding problem

In this section, we connect the embedding problem with commuting and entanglement bias for quantum XOR games. The first step is showing that, when considering Connes' embedding problem, it is enough to consider self-adjoint elements of $B_{qc}(m,m)$ and $B_{qa}(m,m)$ for all $m \geq 2$. Lemma 4.2.1 allows for this reduction.

Lemma 4.2.1. *Let $X \in M_n \otimes M_n$ and $t \in \{qa, qc\}$. Then $X \in B_t(n,n)$ if and only if*

$$W := \begin{pmatrix} 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix} \in B_t(2n,2n).$$

Proof. Suppose that $t = qc$ and $X \in B_{qc}(n,n)$. Then there are unitaries $U = (U_{ij})$, $V = (V_{k\ell}) \in M_n(\mathcal{B}(\mathcal{H}))$ and a vector $\psi \in \mathcal{H}$ of norm 1 such that, for all i,j,k,ℓ, we have $U_{ij}V_{k\ell} = V_{k\ell}U_{ij}$ and

$$X_{(i,j),(k,\ell)} = \langle U_{ij}V_{k\ell}\psi, \psi\rangle.$$

Let $\widetilde{U} = \begin{pmatrix} 0 & (U_{ij}) \\ (U_{ji}^*) & 0 \end{pmatrix} \in M_{2n}(\mathcal{B}(\mathcal{H}))$ and $\widetilde{V} = \begin{pmatrix} 0 & (V_{k\ell}) \\ (V_{\ell k}^*) & 0 \end{pmatrix} \in M_{2n}(\mathcal{B}(\mathcal{H}))$, which are unitary. The entries of $\widetilde{U}$ commute with the entries of $\widetilde{V}$, so with $\widetilde{U}$, $\widetilde{V}$ and ψ, we obtain $W' \in B_{qc}(2n,2n)$, where

$$W' = \begin{pmatrix} 0 & 0 & 0 & X \\ 0 & 0 & \langle U_{ij}V_{\ell k}^*\psi, \psi\rangle & 0 \\ 0 & \langle U_{ji}^*V_{k\ell}\psi, \psi\rangle & 0 & 0 \\ X^* & 0 & 0 & 0 \end{pmatrix}.$$

With $Z = (\langle U_{ij}V_{\ell k}^*\psi, \psi\rangle) \in B_{qc}(n,n)$, we see by Proposition 4.1.3 that

$$W_Z := W' = \begin{bmatrix} 0 & 0 & 0 & X \\ 0 & 0 & Z & 0 \\ 0 & Z^* & 0 & 0 \\ X^* & 0 & 0 & 0 \end{bmatrix} \in B_{qc}(2n,2n).$$

A similar argument using the unitaries $i(U_{ij})$ and $-i(V_{k\ell})$ shows that $W_{-Z} \in B_{qc}(2n,2n)$. By convexity, we obtain $W = \frac{1}{2}(W_Z + W_{-Z}) \in B_{qc}(2n,2n)$. If $t = qa$ and $\varepsilon > 0$, then there is $Y \in B_q(n,n)$ such that $|X_{(i,j),(k,\ell)} - Y_{(i,j),(k,\ell)}| < \varepsilon$ for all i,j,k,ℓ. The above argument shows that

$$R = \begin{bmatrix} 0 & 0 & 0 & Y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y^* & 0 & 0 & 0 \end{bmatrix} \in B_q(2n,2n),$$

and $|R_{(a,b),(c,d)} - W_{(a,b),(c,d)}| < \varepsilon$ for all $1 \le a,b,c,d \le 2n$. Since $B_{qa}(2n,2n) = \overline{B_q(2n,2n)}$, we see that $W \in B_{qa}(2n,2n)$.

Conversely, suppose that

$$W = \begin{bmatrix} 0 & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X^* & 0 & 0 & 0 \end{bmatrix}$$

is in $B_{qa}(2n,2n)$. Let $\varepsilon > 0$; let $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional Hilbert spaces, $U \in M_{2n}(\mathcal{B}(\mathcal{H}_A))$ and $V \in M_{2n}(\mathcal{B}(\mathcal{H}_B))$ be unitaries, and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ be a unit vector such that for all $1 \le a,b,c,d \le 2n$, we have $|W_{(a,b),(c,d)} - \langle U_{ab} \otimes V_{cd}\psi, \psi\rangle| < \varepsilon$. Let

$$S = (U_{i,(j+n)})_{i,j=1}^n \in M_n(\mathcal{B}(\mathcal{H}_A)) \text{ and } T = (V_{k,n+\ell})_{k,\ell=1}^n \in M_n(\mathcal{B}(\mathcal{H}_B)).$$

Then S and T are contractions. Applying the Halmos dilation and performing a canonical shuffle, we obtain unitaries $\widetilde{U} \in M_n(\mathcal{B}(\mathcal{H}_A^{(2)}))$ and $\widetilde{V} \in M_n(\mathcal{B}(\mathcal{H}_B^{(2)}))$, where

$$\widetilde{U}_{ij} = \begin{bmatrix} S_{ij} & (\sqrt{I - S^*S})_{ij} \\ (\sqrt{I - SS^*})_{ij} & -S_{ji}^* \end{bmatrix}$$

and similarly

$$\widetilde{V}_{k\ell} = \begin{bmatrix} T_{k\ell} & (\sqrt{I - T^*T})_{k\ell} \\ (\sqrt{I - TT^*})_{k\ell} & -T_{\ell k}^* \end{bmatrix}.$$

Taking $\widetilde{\psi} = \begin{bmatrix} \psi & 0 & 0 & 0 \end{bmatrix}^t \in \mathcal{H}_A^{(2)} \otimes \mathcal{H}_B^{(2)}$ gives

$$Y = (\langle (S_{ij} \otimes T_{k\ell})\psi, \psi\rangle)_{(i,j),(k,\ell)} = (\langle (\widetilde{U}_{ij} \otimes \widetilde{V}_{k\ell})\widetilde{\psi}, \widetilde{\psi}\rangle)_{(i,j),(k,\ell)} \in B_q(n,n).$$

Moreover, since X is the top right corner block of W, we have $|Y_{(i,j),(k,\ell)} - X_{(i,j),(k,\ell)}| < \varepsilon$ for all i,j,k,ℓ. Since $\varepsilon > 0$ was arbitrary, we conclude that $X \in B_{qa}(n,n)$. Thus, the converse follows for $t = qa$.

Finally, assume that $W \in B_{qc}(2n,2n)$, and let $U, V \in M_{2n}(\mathcal{B}(\mathcal{H}))$ be unitaries and $\psi \in \mathcal{H}$ be a unit vector such that $W_{(a,b),(c,d)} = \langle U_{ab}V_{cd}\psi, \psi\rangle$ for all $1 \le a,b,c,d \le 2n$. As before, let $S = (U_{i,(j+n)})_{i,j=1}^n$ and $T = (V_{k,n+\ell})_{k,\ell=1}^n$. We use an argument similar to the proof of Proposition 2.1.6. First, we let

$$C_{ij} = \begin{bmatrix} S_{ij} & 0 \\ 0 & S_{ij} \end{bmatrix} \in \mathcal{B}(\mathcal{H}^{(2)}) \text{ and } D_{k\ell} = \begin{bmatrix} T_{k\ell} & (\sqrt{I - T^*T})_{k\ell} \\ (\sqrt{I - TT^*})_{k\ell} & -T_{\ell k}^* \end{bmatrix} \in \mathcal{B}(\mathcal{H}^{(2)}).$$

Since the set $\{S_{ij}, S_{ij}^*\}_{i,j=1}^n$ commutes with the set $\{T_{ij}, T_{ij}^*\}_{i,j=1}^n$, it follows that, by examining polynomials in T and T^*, the set $\{S_{ij}, S_{ij}^*\}_{i,j=1}^n$ commutes with each entry of $D_{k\ell}$ and $D_{k\ell}^*$. Therefore, the set $\{C_{ij}, C_{ij}^*\}_{i,j=1}^n$ commutes with the set $\{D_{k\ell}, D_{k\ell}^*\}_{k,\ell=1}^n$, while $C = (C_{ij})$ is a contraction and $D = (D_{k\ell})$ is a unitary. Performing a similar dilation on C and replacing $D_{k\ell}$ with $\begin{bmatrix} D_{k\ell} & 0 \\ 0 & D_{k\ell} \end{bmatrix}$, we obtain unitaries $A = (A_{ij})$ and $B = (B_{k\ell})$ in $M_n(\mathcal{B}(\mathcal{H}^{(4)}))$ such that the $(1,1)$-block of A_{ij} is S_{ij} and the $(1,1)$-block of $B_{k\ell}$ is $T_{k\ell}$. Letting $\widetilde{\psi} = \begin{bmatrix} \psi & 0 & 0 & 0 \end{bmatrix}^t \in \mathcal{H}^{(4)}$, we see that

$$X = (\langle A_{ij} B_{k\ell} \widetilde{\psi}, \widetilde{\psi} \rangle)_{(i,j),(k,\ell)} \in B_{qc}(n,n),$$

which completes the proof. □

The following theorem shows that it is enough to consider self-adjoint elements of $M_n \otimes M_n$ for the embedding problem.

Theorem 4.2.2. *The following are equivalent.*

1. *Connes' Embedding Problem has a positive answer.*
2. $B_{qa}(n,n) = B_{qc}(n,n)$ *for all* $n \geq 2$.
3. $M_n \otimes_{qa} M_n = M_n \otimes_{qc} M_n$ *isometrically for all* $n \geq 2$.
4. *For every* $n \geq 2$ *and* $X = X^* \in M_n \otimes M_n$ *with* $\|X\|_{qc} = \|X\|_{M_n \otimes_{\min} M_n} = 1$*, we have* $\|X\|_{qa} = 1$.
5. *For every* $n \geq 2$ *and* $M = M^* \in M_n \otimes M_n$ *with* $\|M\|_{M_n \otimes_{\min} M_n}^* = 1$*, we have* $\|M\|_{qa}^* = \|M\|_{qc}^*$.

Proof. The equivalence of (1), (2) and (3) is by Theorem 3.2.7. Clearly (3) implies (4) and (5). We will show that (5) implies (2); the proof that (4) implies (2) is similar. Let $A \in M_n \otimes M_n$. By the proof of Theorem 3.2.7, it suffices to know that the following holds for all $n \geq 2$: whenever $Y \in B_{qc}(n,n)$ is diagonal with diagonal entries $\tau(u_i u_j^*)$ for some unitaries $u_1, ..., u_n$ in a unital C^*-algebra $\mathcal{A}$ and a tracial state τ on $\mathcal{A}$, we have that $Y \in B_{qa}(n,n)$. In particular, there are entries in such Y equal to 1. Therefore, if $\|\cdot\|$ denotes the operator norm on $M_n \otimes M_n$, then

$$1 \le \|Y\| \le \|Y\|_{qc} = 1.$$

Now, assume that (2) fails. Then there is $Y \in B_{qc}(n,n)$ with operator norm 1 such that $Y \notin B_{qa}(n,n)$. By Lemma 4.2.1, there is $n \ge 2$ and some $X = X^* \in M_{2n} \otimes M_{2n}$ such that $\|X\|_{qc} = \|X\| = 1$ but $\|X\|_{qa} > 1$. By the Hahn-Banach Theorem, we have

$$\begin{aligned} 1 &= \sup\{|g(X)| : g \in (M_{2n} \otimes M_{2n})^*,\ \|g\|^*_{M_n \otimes_{\min} M_n} = 1\} \\ &< \sup\{|g(X)| : g \in (M_{2n} \otimes M_{2n})^*,\ \|g\|^*_{qa} = 1\}. \end{aligned}$$

Therefore, there is $g \in (M_{2n} \otimes M_{2n})^*$ such that $\|g\|^*_{qa} = 1$ but $|g(X)| = \alpha > 1$. By multiplying g by some $z \in \mathbb{T}$ if necessary, we may assume that $g(X) = \alpha$. Let $M \in M_{2n} \otimes M_{2n}$ be the matrix such that $g(Y) = \mathrm{Tr}(YM)$ for all $Y \in M_{2n} \otimes M_{2n}$. Since $X = X^*$, we see that

$$g^*(X) = \mathrm{Tr}(XM^*) = \overline{\mathrm{Tr}(MX)} = \overline{\mathrm{Tr}(XM)} = \overline{g(X)} = \alpha.$$

Since $\|\cdot\|^*_{qa}$ is a $*$-norm, we obtain $\|g^*\|^*_{qa} = 1$. Thus, $f = \mathrm{Re}(g) = \frac{g+g^*}{2}$ is a functional with

$$\|f\|^*_{qa} \le 1 < f(X) = \alpha \le \|f\|^*_{M_n \otimes_{\min} M_n}.$$

Dividing by $\|f\|^*_{M_n \otimes_{\min} M_n}$ if necessary, we may assume that $\|f\|^*_{M_n \otimes M_n} = 1$, while $\|f\|^*_{qa} \le 1 - \varepsilon$ for some $\varepsilon > 0$ and $f(X) > 1 - \varepsilon$. The associated matrix to f is $\frac{M+M^*}{2}$, which is self-adjoint. By the contrapositive, (5) implies (2). □

As a corollary, we can describe the embedding problem in terms of optimal strategies for quantum XOR games.

Corollary 4.2.3. *Connes' embedding problem has a positive answer if and only if, for every $n \ge 2$ and for every quantum XOR game G of size n with associated matrix $M \in M_n \otimes M_n$, we have $\omega^*_{qa}(G) = \omega^*_{qc}(G)$.*

Proof. For $t \in \{qa, qc\}$, the quantity $\omega^*_t(G)$ is the norm of a self-adjoint linear functional on $M_n \otimes_t M_n$. In particular, if Connes' embedding problem holds, then $M_n \otimes_{qa} M_n = M_n \otimes_{qc} M_n$ isometrically for all n, so that $\omega^*_{qa}(G) = \omega^*_{qc}(G)$ for all quantum XOR games G. Conversely, if Connes' embedding problem has a negative answer, then by Theorem 4.2.2, there is some $n \geq 2$, $M = M^* \in M_n \otimes M_n$, and $0 < \varepsilon < 1$ such that

$$1 - \varepsilon = \|M\|_{qa}^* < \|M\|_{qc}^* \leq \|M\|_{M_n \otimes_{\min} M_n}^* = 1.$$

This implies that $\|M\|_1 = 1$. Then by Proposition 4.1.1, there is a quantum XOR game G with associated matrix M. By the choice of M, for every correlation $X = (X_{(i,j),(k,\ell)})_{(i,j),(k,\ell)} \in B_{qa}(n,n)$, we have

$$\omega_{qa}^*(G; X) \leq 1 - \varepsilon,$$

which implies that $\omega_{qa}^*(G) \leq 1 - \varepsilon$. Meanwhile, there is $Y \in B_{qc}(n,n)$ with $\omega_{qc}^*(G; Y) > 1 - \varepsilon$, so that $\omega_{qc}^*(G) > \omega_{qa}^*(G)$. This completes the proof. ☐

Initially, we thought that it sufficed in Corollary 4.2.3 to consider the case when $\omega_{qc}^*(G) = 1$. If this were the case, then one would only need to consider quantum XOR games for which there was a perfect commuting strategy. Nevertheless, we are unsure if this holds.

Problem 4.2.4. *Is Connes' embedding problem equivalent to the assertion that every quantum XOR game G with winning commuting strategy has a winning approximate finite-dimensional strategy?*

Chapter 5

Quantum teleportation and super-dense coding in operator algebras

Matrix-valued correlations can be thought of as the outcomes of partial measurements, and from the C^*-algebraic perspective, $C_{qs}^{(n)}(m,k)$ is the natural generalization of $C_{qs}(m,k)$ obtained by replacing states by M_n-valued ucp maps. The study of the sets of matrix-valued quantum correlations led to the equivalence of Connes' embedding problem and a matrix-valued version of Tsirelson's problem (see [34, 25]). Later, Ozawa [47] proved that the scalar version of the Tsirelson problem is equivalent to Connes' embedding problem. In this chapter, we exhibit non-spatial matrix-valued correlations for the smallest non-trivial input and output sets. In particular, we show the following:

Theorem 5.0.5. *There exists an $n \leq 13$ such that, whenever $m,k \in \mathbb{N}$ are such that $m,k \geq 2$ and $(m,k) \neq (2,2)$, then $C_{qs}^{(n)}(m,k) \neq C_{qa}^{(n)}(m,k)$. In particular, $C_{qs}^{(n)}(m,k)$ is not closed.*

We will obtain Theorem 5.0.5 by showing that $C_{qs}^{(5)}(3,2)$, $C_{qs}^{(3)}(4,2)$ and $C_{qs}^{(13)}(2,3)$ are not closed. Theorem 5.0.5 follows easily from the non-closure of $C_{qs}^{(5)}(3,2)$ and $C_{qs}^{(13)}(2,3)$ with an application of Proposition 1.8.7.

Our methods draw on two interesting connections between quantum information theory and operator algebras. The first one is a reformulation of **super-dense coding** [5] and **teleportation** [4] in terms of isomorphisms of certain C^*-algebras. Super-dense coding and teleportation are two fundamental protocols discovered by Bennett and his coauthors in the early age of modern quantum information theory. These two protocols together describe the fact that, with the assistance of quantum entanglement, quantum communication and classical communication are mutually convertible resources [1, 17], in the sense that the one can be reduced to the other (and vice versa). Indeed, suppose that Alice and Bob share a maximally entangled state φ on their resource space $\mathcal{H}_A \otimes \mathcal{H}_B$. Then quantum teleportation allows Alice to effectively send a quantum state to Bob by only sending a finite amount of classical information, in the presence of the state φ. If Alice wants to send her state ρ from her state space to Bob's state space, she can perform a measurement on her state space and her part of the resource space with respect to a certain basis, and send the (classical) outcome to Bob. Bob can then decode this outcome by applying a unitary transformation on his part of the resource space and his state space to receive Alice's state on his state space. For super-dense coding, the presence of φ allows Alice to send two bits to Bob at once, by sending her part of a modified maximally entangled state to Bob using a (noiseless) quantum channel. This modified maximally entangled state is obtained by a unitary operation on Alice's state space and her part of the resource space. Bob can then obtain Alice's classical information by a measurement on his part of the resource space and his state space with respect to a certain basis. The protocols of quantum teleportation and super-dense coding are examples of the extraordinary power of entanglement, and they demonstrate the fundamental role of non-local correlations in quantum information science.

We will show that the protocol maps of super-dense coding and teleportation translate into the following C^*-algebra isomorphisms:

Theorem 5.0.6. *With certain actions of $\mathbb{Z}_d$, we have*

$$M_d(C^*(\mathbb{F}_{d^2})) \cong \mathcal{U}_{nc}(d) \rtimes \mathbb{Z}_d \rtimes \mathbb{Z}_d \ , \ \text{and } M_d(\mathcal{U}_{nc}(d)) \cong C^*(\mathbb{F}_{d^2}) \rtimes \mathbb{Z}_d \rtimes \mathbb{Z}_d.$$

As a consequence, $\mathcal{U}_{nc}(d)$ (respectively, $C^*(\mathbb{F}_{d^2})$) is a C^*-subalgebra of $M_d(C^*(\mathbb{F}_{d^2}))$ (respectively, $M_d(\mathcal{U}_{nc}(d))$) with a faithful conditional expectation onto it.

The operator space perspective of super-dense coding and teleportation has been studied in [35]. In particular, by [35, Corollary 1.2 & Theorem 1.3], the trace class S_1^d and ℓ_1-sequence space $\ell_1^{d^2}$, equipped with their natural operator space structures $S_1^d = (M_d)^*$ and $\ell_1^{d^2} = (\ell_\infty^{d^2})^*$, embed into certain matrix levels of each other via complete isometries; i.e.,

$$S_1^d \hookrightarrow M_d(\ell_1^{d^2}) \ , \ \text{and } \ \ell_1^{d^2} \hookrightarrow M_d(S_1^d) \ . \tag{5.0.1}$$

We will see that $C^*(\mathbb{F}_{d^2})$ (respectively, $\mathcal{U}_{nc}(d)$) is the C^*-envelope of $\ell_1^{d^2}$ (respectively, S_1^d) using suitable unitizations. Theorem 5.0.6 can be viewed as liftings of the embeddings from (5.0.1) to the level of C^*-algebras. It provides explicit connections between the two universal C^*-algebras and relates to the unitary correlation sets from Chapter 3.

The second ingredient in our argument towards Theorem 5.0.5 is embezzlement of entanglement. This phenomenon was used in Chapter 3 to separate the unitary correlation sets in the qs and qa models. This separation corresponds to a state on the minimal tensor product $\mathcal{U}_{nc}(2)\otimes_{min}\mathcal{U}_{nc}(2)$ that cannot be implemented as a vector state via tensor product representations. Our idea is to apply the $*$-isomorphisms in Theorem 5.0.6 to translate the non-spatial correlation from $\mathcal{U}_{nc}(2) \otimes_{min} \mathcal{U}_{nc}(2)$ to $M_2(C^*(\mathbb{F}_4)) \otimes_{min} M_2(C^*(\mathbb{F}_4))$, and then use group embeddings of free groups into free products of cyclic groups to obtain non-spatial matrix-valued correlations. Our method is the first attempt to use embezzlement to prove a statement within the range of Tsirelson's problem.

This chapter is organized as follows. In Section 5.1, we review the basics of the protocols of super-dense coding and teleportation, and give the proof of Theorem 5.0.6. Based on that section, we show the non-closure of the matrix-valued correlation sets $C_{qs}^{(5)}(3,2)$, $C_{qs}^{(3)}(4,2)$ and $C_{qs}^{(13)}(2,3)$ in Section 5.2.

5.1 Teleportation and super-dense coding

We briefly review the basic protocols of teleportation and super-dense coding and refer to [68] for their information-theoretic meaning. Let M_d be the space of $d \times d$ complex matrices, and let ℓ_2^d be the d-dimensional complex Hilbert space. We let $\{e_j\}_{j=0}^{d-1}$ be the standard basis of ℓ_2^d and denote by $\{E_{jk}\}_{0\le j,k\le d-1}$ the standard matrix units of M_d given by $E_{jk} = e_j e_k^*$. The maximally entangled state on $\ell_2^d \otimes \ell_2^d$ is

$$\phi = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} e_j \otimes e_j,$$

and its corresponding density matrix is the rank-one projection

$$\phi\phi^* = \frac{1}{d} \sum_{j,k=0}^{d-1} E_{jk} \otimes E_{jk}.$$

The generalized Pauli matrices are given by

$$Xe_j = e^{\frac{2\pi i j}{d}} e_j \, , \; Ze_j = e_{j+1} \, , \; \forall\, 0 \le j \le d-1.$$

In the definition of Z and in the remainder of the chapter, the addition of indices will be considered modulo d. It is helpful to note that $XZ = e^{\frac{2\pi i}{d}} ZX$.

For each $0 \leq j, k \leq d-1$, we introduce the operator $T_{j,k} := X^j Z^k$ and the vector

$$\phi_{jk} := (T_{j,k} \otimes 1)\phi = \frac{1}{\sqrt{d}} \sum_{\ell=0}^{d-1} e^{\frac{2\pi i j(\ell+k)}{d}} e_{k+\ell} \otimes e_\ell.$$

The density matrix associated with ϕ_{jk} is the rank-one projection

$$\phi_{jk}\phi_{jk}^* = \frac{1}{d} \sum_{\ell,n=0}^{d-1} e^{\frac{2\pi i(\ell-n)j}{d}} E_{k+\ell,k+n} \otimes E_{\ell n}.$$

We observe that the set $\{T_{j,k} : 0 \leq j, k \leq d-1\}$ forms an orthonormal basis for M_d with respect to the Hilbert-Schmidt norm. Indeed, we first observe that

$$T_{j_1,k_1}^* T_{j_2,k_2} = Z^{-k_1} X^{j_2-j_1} Z^{k_2} = \exp\left(\frac{2\pi i(j_2-j_1)}{d}\right) T_{j_2-j_1,k_2-k_1}.$$

Evidently, we have $\operatorname{tr}(T_{j,k}) = 0$ whenever $(j,k) \neq (0,0)$, where tr is the normalized trace on M_d. Therefore, $\operatorname{tr}(T_{j_1,k_1}^* T_{j_2,k_2}) = \delta_{j_1,j_2}\delta_{k_1,k_2}$, yielding the desired claim. Similarly, we note that $\{\phi_{jk}\}_{0\leq j,k\leq d-1}$ is a set of maximally entangled vectors, and they form an orthonormal basis for $\ell_2^d \otimes \ell_2^d$.

Mathematically, the protocol of quantum teleportation for a d-dimensional system can be expressed as follows:

$$\begin{array}{ccccccc} M_d & \longrightarrow & M_d \otimes M_d \otimes M_d & \longrightarrow & \ell_\infty^{d^2} \otimes M_d & \longrightarrow & M_d \\ \rho & \longmapsto & \rho \otimes \phi\phi^* & \longmapsto & \frac{1}{d^2}\sum_{j,k=0}^{d-1} e_{jk} \otimes T_{jk}^* \rho T_{jk} & \longmapsto & \rho, \end{array} \tag{5.1.1}$$

where $\{e_{jk}\}_{j,k=0}^{d-1}$ denotes the standard orthonormal basis for $\ell_\infty^{d^2}$. Here, the matrix ρ can be thought of as the quantum state that the sender Alice sends to the receiver Bob. In the protocol, Alice first performs a measurement according to the basis $\{\phi_{jk}\}_{j,k=0}^{d-1}$ on the coupled system of the input ρ and her part of the maximally entangled state ϕ. She sends the outcome of her measurement, a classical signal of cardinality d^2, to Bob via some classical channel. Then Bob reproduces the state ρ by doing a unitary operation on his part according to the information received from Alice. Here a key calculation (see [35, Lemma 2.1]) is that

$$\rho \otimes \phi\phi^* = \frac{1}{d^2} \sum_{j,k,j',k'=0}^{d-1} \phi_{jk}\phi_{j'k'}^* \otimes T_{jk}^* \rho T_{j'k'} \ .$$

The second map of (5.1.1), which corresponds to the measurement performed by Alice, is the conditional expectation from $M_d \otimes M_d$ onto the commutative subalgebra spanned by $\{\phi_{jk}\phi_{jk}^*\}_{j,k}$. Since this subalgebra is commutative and d^2-dimensional, it may be identified with $\ell_\infty^{d^2}$, as it is in the second map of (5.1.1). Bob's action is the third map, which is of the form

$$\sum_{j,k=0}^{d-1} p_{jk} e_{jk} \otimes \rho_{jk} \mapsto \sum_{0 \le j,k \le d-1} p_{jk} T_{jk}^* \rho_{jk} T_{jk} \ ,$$

where each $\rho_{jk} \in M_d$. Using the same notation, super-dense coding can be expressed as follows:

$$\begin{array}{ccccccc} \ell_\infty^{d^2} & \longrightarrow & \ell_\infty^{d^2} \otimes M_d \otimes M_d & \longrightarrow & M_d \otimes M_d & \longrightarrow & \ell_\infty^{d^2} \\ & & & & & & \\ (p_{jk}) & \longmapsto & \left(\sum_{j,k=0}^{d-1} p_{jk} e_{jk}\right) \otimes \phi\phi^* & \longmapsto & \sum_{jk} p_{jk} \phi_{jk}\phi_{jk}^* & \longmapsto & (p_{jk}). \end{array} \tag{5.1.2}$$

This time Alice wants to transmit a classical signal (p_{jk}), which is a probability distribution. She first applies the unitary T_{jk} on her part of the maximally entangled state ϕ according to the signal (p_{jk}), and then sends her part of ϕ to Bob via some quantum channel. Now Bob has both parts of the (modified) entangled state, and can perfectly decode the classical signal (p_{jk}) via a measurement according to the basis $\{\phi_{jk}\}_{j,k=0}^{d-1}$.

With respect to certain operator space structures, the completely bounded norms of the above maps were calculated in [35]. Recall that the natural operator space structures of S_1^d and ℓ_1^d are given by the operator space duality $S_1^d = (M_d)^*$ and $\ell_1^d = (\ell_\infty^d)^*$, with matrix norms given as follows: for $n \in \mathbb{N}$ and $\sum_{j=0}^{d-1} A_j \otimes e_j \in M_n(\ell_1^d)$ and $\sum_{j,k=0}^{d-1} A_{jk} \otimes E_{jk} \in$

$M_n(S_1^d)$, we have

$$\left\|\sum_{j=0}^{d-1} A_j \otimes e_j\right\| = \sup\left\{\left\|\sum_{j=0}^{d-1} A_j \otimes B_j\right\| \ : B_j \in \mathcal{B}(\mathcal{H}),\ \|B_j\| \leq 1\right\} ,$$
$$\left\|\sum_{j,k=0}^{d-1} A_{jk} \otimes E_{jk}\right\| = \sup\left\{\left\|\sum_{j,k=0}^{d-1} A_{jk} \otimes B_{jk}\right\| \ : B_{jk} \in \mathcal{B}(\mathcal{H}),\ \|(B_{jk})\|_{M_d(\mathcal{B}(\mathcal{H}))} \leq 1\right\} .$$

Regarding the protocols of teleportation and super-dense coding, we have the following theorem:

Theorem 5.1.1. (Junge-Palazuelos, [35, Section 2]) *The following maps are completely contractive:*

$$\mathcal{L}_1 : S_1^d \to M_d(\ell_1^{d^2}) \ , \ \mathcal{L}_1(\rho) = \frac{1}{d}\sum_{j,k=0}^{d-1} T_{jk}^* \rho T_{jk} \otimes e_{jk} \ ,$$
$$\mathcal{N}_1 : M_d(\ell_1^{d^2}) \to S_1^d \ , \ \mathcal{N}_1(\rho \otimes e_{jk}) = \frac{1}{d} T_{jk} \rho T_{jk}^* \ ,$$
$$\mathcal{L}_2 : \ell_1^{d^2} \to M_d(S_1^d) \ , \ \mathcal{L}_2\left(\sum_{j,k=0}^{d-1} p_{jk} e_{jk}\right) = d\sum_{j,k=0}^{d-1} p_{jk} \phi_{j,d-k} \phi_{j,d-k}^* \ ,$$
$$\mathcal{N}_2 : M_d(S_1^d) \to \ell_1^{d^2} \ , \ \mathcal{N}_2(\rho) = \frac{1}{d}\sum_{j,k=0}^{d-1} tr(\rho \phi_{j,d-k} \phi_{j,d-k}^*) e_{jk} \ .$$

Moreover, $\mathcal{N}_1 \circ \mathcal{L}_1 = id_{S_1^d}$ *and* $\mathcal{N}_2 \circ \mathcal{L}_2 = id_{\ell_1^{d^2}}$. *In particular,* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are complete isometries.*

Remark 5.1.2. The above complete contractions differ from the trace preserving maps in (5.1.1) and (5.1.2) by a scaling constant d. This difference is because, for each of the maximally entangled vectors ϕ_{jk}, the corresponding density matrix satisfies

$$\| \phi_{jk} \phi_{jk}^* \|_{S_1^{d^2}} = 1 \ , \ \| \phi_{jk} \phi_{jk}^* \|_{M_d(S_1^d)} = \frac{1}{d}.$$

Indeed, the first norm is the trace norm of $\phi_{jk}\phi_{jk}^*$, which is 1 since $\phi_{jk}\phi_{jk}^*$ is positive with trace 1. For the second norm, we observe that

$$\phi_{jk}\phi_{jk}^* = (T_{j,k} \otimes I_d)\phi\phi^*(T_{j,k}^* \otimes I_d).$$

It is not hard to see that $\|\phi_{jk}\phi_{jk}^*\|_{M_d(S_1^d)} = \|\phi\phi^*\|_{M_d(S_1^d)}$. The linear map from M_d to M_d that is associated with $\phi\phi^*$ is $\frac{1}{d}\mathrm{id}_{M_d}$. Considering the operator space structure of $M_d(S_1^d)$, we obtain $\|\phi\phi^*\|_{M_d(S_1^d)} = \frac{1}{d}$, yielding the second norm.

In addition to the scaling constant d, we also have flipped the indices $(j,k) \to (j, d-k)$ in $\mathcal{L}_2$ and $\mathcal{N}_2$; however, it is clear that our protocol is equivalent to the original protocol.

Our candidates for a C^*-algebraic analogue of teleportation and super-dense coding are the "smallest" C^*-algebras containing S_1^d and $\ell_1^{d^2}$ respectively. The correct notion of smallness is the C^*-envelope. We recall that a (concrete) unital operator space E is a closed subspace of a C^*-algebra containing the identity. The C^*-envelope $C^*_{env}(E)$ of a unital operator space E is the unique C^*-algebra $C^*_{env}(E)$ equipped with a unital complete isometry $\iota : E \to C^*_{env}(E)$ satisfying the following property: for any unital complete isometry $j : E \to B(H)$, there exists a unique surjective $*$-homomorphism $\pi : C^*(j(E)) \to C^*_{env}(E)$ such that $\pi \circ j = \iota$, where $C^*(j(E))$ is the C^*-subalgebra of $B(H)$ generated by the image $j(E)$.

The following proposition gives the C^*-algebras needed for a C^*-algebraic analogue of the protocols of quantum teleportation and super-dense coding.

Proposition 5.1.3. *Let $\{g_j\}_{j=0}^{d-1}$ be the generators of $C^*(\mathbb{F}_d)$, and let $\{u_{jk}\}_{j,k=0}^{d-1}$ be the generators of $\mathcal{U}_{nc}(d)$. Define the operator spaces and unitizations*

$$\mathcal{F}_d := span\ (\{g_j\}_{j=0}^{d-1}) \subseteq C^*(\mathbb{F}_d)\ ,\ \widetilde{\mathcal{F}}_d = span\ (1 \cup \mathcal{F}_d) \subseteq C^*(\mathbb{F}_d)\ ;$$
$$\mathcal{G}_d := span\ (\{u_{jk}\}_{j,k=0}^{d-1}) \subseteq \mathcal{U}_{nc}(d)\ ,\ \widetilde{\mathcal{G}}_d = span\ (1 \cup \mathcal{G}_d) \subseteq \mathcal{U}_{nc}(d)\ .$$

Then:

i) $\mathcal{F}_d \cong \ell_1^d$ completely isometrically and $C^*_{env}(\widetilde{\mathcal{F}}_d) \cong C^*(\mathbb{F}_d)$.

ii) $\mathcal{G}_d \cong S_1^d$ completely isometrically and $C^*_{env}(\widetilde{\mathcal{G}}_d) \cong \mathcal{U}_{nc}(d)$.

Proof. The complete isometry in i) is from [70]; we include the proof for completeness. Let $\mathcal{H}$ be an infinite dimensional Hilbert space and A_j, A_{jk} be matrices in M_n for $0 \leq j,k \leq$

$d-1$. For an element $\sum_{j=0}^{d-1} A_j \otimes e_j \in M_n(\ell_1^d)$, we have

$$\begin{aligned}\left\|\sum_{j=0}^{d-1} A_j \otimes e_j\right\| &= \sup\left\{\left\|\sum_{j=0}^{d-1} A_j \otimes B_j\right\| \ : B_j \in \mathcal{B}(\mathcal{H}),\ \|B_j\| \leq 1\right\} \\ &\geq \sup\left\{\left\|\sum_{j=0}^{d-1} A_j \otimes C_j\right\| \ : C_j \text{ unitary in } \mathcal{B}(\mathcal{H})\right\} \\ &= \left\|\sum_{j=0}^{d-1} A_j \otimes g_j\right\|_{M_n(C^*(\mathbb{F}_d))}.\end{aligned}$$

To show the reverse inequality, we let $B_1, ..., B_{d-1} \in \mathcal{B}(\mathcal{H})$ be contractions. Then

$$C_j := \begin{pmatrix} B_j & \sqrt{I - B_j B_j^*} \\ \sqrt{I - B_j^* B_j} & -B_j^* \end{pmatrix} \in M_2(\mathcal{B}(\mathcal{H}))$$

is unitary for each j. Applying a compression to the $(1,1)$ corner shows that

$$\left\|\sum_{j=0}^{d-1} A_j \otimes B_j\right\|_{M_n(\mathcal{B}(\mathcal{H}))} \leq \left\|\sum_{j=0}^{d-1} A_j \otimes C_j\right\|_{M_n(M_2(\mathcal{B}(\mathcal{H})))}.$$

Since $B_1, ..., B_{d-1}$ are contractions and $C_1, ..., C_{d-1}$ are unitaries, it follows that

$$\begin{aligned}\left\|\sum_{j=0}^{d-1} A_j \otimes e_j\right\|_{M_n(\ell_1^d)} &\leq \sup\left\{\left\|\sum_{j=0}^{d-1} A_j \otimes C_j\right\| : C_j \text{ unitary in } \mathcal{B}(\mathcal{H})\right\} \\ &= \left\|\sum_{j=0}^{d-1} A_j \otimes g_j\right\|_{M_n(C^*(\mathbb{F}_d))}.\end{aligned}$$

Similarly, if $\sum_{j,k=0}^{d-1} A_{jk} \otimes E_{jk}$ is an element of $M_n(S_1^d)$, then

$$\begin{aligned}
\left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes E_{jk} \right\| &= \sup \left\{ \left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes B_{jk} \right\| : B_{jk} \in \mathcal{B}(\mathcal{H}),\ \|(B_{jk})\|_{M_d(\mathcal{B}(\mathcal{H}))} \leq 1 \right\} \\
&\geq \sup \left\{ \left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes C_{jk} \right\| : (C_{jk})_{j,k=0}^{d-1} \text{ unitary in } M_d(\mathcal{B}(\mathcal{H})) \right\} \\
&= \left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes u_{jk} \right\|_{M_n(\mathcal{U}_{nc}(d))} .
\end{aligned}$$

For the reverse inequality, let $(B_{jk}) \in M_d(\mathcal{B}(\mathcal{H}))$ be a contraction. By Proposition 2.1.1, there is a matrix $(C_{jk}) \in M_d(M_2(\mathcal{B}(\mathcal{H})))$ that is unitary, such that the $(1,1)$ corner of each C_{jk} is B_{jk}. It follows that

$$\left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes B_{jk} \right\| \leq \left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes C_{jk} \right\| .$$

Taking the respective suprema, we obtain

$$\left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes E_{jk} \right\|_{M_n(S_1^d)} \leq \left\| \sum_{j,k=0}^{d-1} A_{jk} \otimes u_{jk} \right\|_{M_n(\mathcal{U}_{nc}(d))} ,$$

establishing the reverse inequality.

Thus, the maps

$$\begin{aligned}
j_1 : \ell_1^d \to \mathcal{F}_d \subset C^*(\mathbb{F}_d) \,, \quad j_1(e_j) = g_j \,, \\
j_2 : S_1^d \to \mathcal{G}_d \subset \mathcal{U}_{nc}(d) \,, \quad j_2(E_{jk}) = u_{jk} \,,
\end{aligned}$$

are complete isometries.

We now show that $C^*_{env}(\widetilde{\mathcal{F}}_d) \cong C^*(\mathbb{F}_d)$. Since there is a unital completely isometric inclusion $\widetilde{\mathcal{F}}_d \subseteq C^*(\mathbb{F}_d)$, by definition of the C^*-envelope, there is a surjective $*$-homomorphism $\gamma : C^*(\mathbb{F}_d) \to C^*_{env}(\widetilde{\mathcal{F}}_d)$ such that $\gamma(g_j) = e_j$ for all $0 \leq j \leq d-1$. Since each g_j is unitary in $C^*(\mathbb{F}_d)$, each e_j is unitary in $C^*_{env}(\widetilde{\mathcal{F}}_d)$. Now, we may assume

assume that $C^*(\mathbb{F}_d) \subseteq \mathcal{B}(\mathcal{H})$ is faithfully represented on some Hilbert space $\mathcal{H}$. By Wittstock's extension theorem [69], the unital complete isometry $\eta : \widetilde{\mathcal{F}}_d \to C^*(\mathbb{F}_d) \subseteq B(H)$ given by $\eta(e_j) = g_j$ extends to a unital completely contractive (hence completely positive) map from $C^*_{env}(\widetilde{\mathcal{F}}_d)$ to $B(H)$, which we will also denote by η. Choose a minimal Stinespring representation $\eta(\cdot) = V^*\pi(\cdot)V$ for η on some Hilbert space H_π. Since η is unital, V is an isometry. We may write $H_\pi = ran(V) \oplus ran(V)^\perp$. With respect to this decomposition,

$$\pi(e_j) = \begin{bmatrix} \eta(e_j) & * \\ * & * \end{bmatrix} = \begin{bmatrix} g_j & * \\ * & * \end{bmatrix} .$$

Since e_j is unitary in $C^*_{env}(\widetilde{\mathcal{F}}_d)$, $\pi(e_j)$ must be unitary in $B(H_\pi)$. But the $(1,1)$ entry of $\pi(e_j)$ is unitary as well, so the $(1,2)$ and $(2,1)$ entries must be 0. Therefore,

$$\pi(e_j) = \begin{bmatrix} g_j & 0 \\ 0 & * \end{bmatrix} = \begin{bmatrix} \eta(e_j) & 0 \\ 0 & * \end{bmatrix} .$$

Thus, η is multiplicative on the generating set $\{e_j\}_{j=0}^{d-1}$ for $C^*_{env}(\widetilde{\mathcal{F}}_d)$, so η must be a $*$-homomorphism. Moreover, η is surjective onto $C^*(\mathbb{F}_d)$ because the set $\{g_j\}_{0\le j\le d-1}$ generates $C^*(\mathbb{F}_d)$. Since $\eta \circ \gamma(g_j) = g_j$ and $\gamma \circ \eta(e_j) = e_j$ for all j, it follows that $\eta \circ \gamma = \mathrm{id}_{C^*(\mathbb{F}_d)}$ and $\gamma \circ \eta = \mathrm{id}_{C^*_{env}(\widetilde{\mathcal{F}}_d)}$. Hence, $C^*_{env}(\widetilde{\mathcal{F}}_d)$ is isomorphic to $C^*(\mathbb{F}_d)$. The proof that $C^*_{env}(\widetilde{\mathcal{G}}_d) = \mathcal{U}_{nc}(d)$ is identical to part (2) of Theorem 2.1.3. □

The next lemma shows that the embeddings from Theorem 5.1.1 can be extended to $*$-homomorphisms on the respective C^*-envelopes. We will denote these $*$-homomorphisms by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.

Lemma 5.1.4. *Let $\{g_{\ell m}\}_{0\le \ell,m\le d-1}$ be a set of universal generators for $C^*(\mathbb{F}_{d^2})$, and let $\{u_{jk}\}_{0\le j,k\le d-1}$ be the generators of $\mathcal{U}_{nc}(d)$. Define $\mathcal{L}_1 : \mathcal{U}_{nc}(d) \to M_d(C^*(\mathbb{F}_{d^2}))$ and $\mathcal{L}_2 : C^*(\mathbb{F}_{d^2}) \to M_d(\mathcal{U}_{nc}(d))$ by*

$$\mathcal{L}_1(u_{jk}) = \frac{1}{d}\sum_{\ell,m=0}^{d-1} e^{-\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes g_{\ell m} ,$$

$$\mathcal{L}_2(g_{\ell m}) = \sum_{j,k=0}^{d-1} e^{\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes u_{jk} .$$

Then both $\mathcal{L}_1$ and $\mathcal{L}_2$ extend to $*$-homomorphisms.

Proof. Let $U = \sum_{j,k=0}^{d-1} E_{jk} \otimes u_{jk} \in M_d(\mathcal{U}_{nc}(d))$ be the fundamental unitary of $\mathcal{U}_{nc}(d)$. Note that

$$\sum_{j,k} E_{jk} \otimes \mathcal{L}_1(u_{jk}) = \frac{1}{d} \sum_{j,k,\ell,m} E_{jk} \otimes e^{-\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes g_{\ell m} = \sum_{\ell,m} \phi_{-\ell,m}\phi^*_{-\ell,m} \otimes g_{\ell m}$$

is a unitary in $M_d \otimes M_d(C^*(\mathbb{F}_{d^2}))$. By the universal property of $\mathcal{U}_{nc}(d)$, $\mathcal{L}_1$ is a unital $*$-homomorphism. For the second map $\mathcal{L}_2$, we observe that

$$\mathcal{L}_2(g_{\ell m}) = \sum_{j,k} e^{\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes u_{jk} = (T_{\ell,-m} \otimes 1)U(T_{\ell,-m} \otimes 1)^*$$

is a unitary in $M_d(\mathcal{U}_{nc}(d))$ for each $0 \le \ell, m \le d-1$, since $T_{\ell,-m}$ is unitary. By the universal property of $C^*(\mathbb{F}_{d^2})$, $\mathcal{L}_2$ is a unital $*$-homomorphism, as desired. $\square$

Moving towards a proof of Theorem 5.0.6, we will consider two automorphisms α_1, α_2 of $\mathcal{U}_{nc}(d)$ given as follows:

$$\alpha_1(u_{jk}) = e^{\frac{2\pi i(j-k)}{d}} u_{jk} \ , \ \alpha_2(u_{jk}) = u_{j+1,k+1}, \ \forall\, 0 \le j,k \le d-1,$$

where, in the definition of α_2, the addition of indices is done modulo d. It is not hard to check that $(\alpha_1(u_{jk}))$ and $(\alpha_2(u_{jk}))$ are unitary in $M_d(\mathcal{U}_{nc}(d))$, so that each $\alpha_i : \mathcal{U}_{nc}(d) \to \mathcal{U}_{nc}(d)$ is a unital $*$-homomorphism. Both α_1 and α_2 have order d; that is, $\alpha_1^d = \alpha_2^d = \mathrm{id}_{\mathcal{U}_{nc}(d)}$ and $\alpha_i^k \neq \mathrm{id}_{\mathcal{U}_{nc}(d)}$ for all $1 \le k \le d-1$ and $i = 1, 2$. Therefore, $\alpha_1, \alpha_2 \in \mathrm{Aut}(\mathcal{U}_{nc}(d))$. The automorphisms α_1 and α_2 give two actions of the cyclic group $\mathbb{Z}_d$ on $\mathcal{U}_{nc}(d)$. We may extend the definition of α_2 to an automomorphism of $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d$, as follows: if v is the generator of $\mathbb{Z}_d$ in $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d$, then we set $\alpha_2(v) = e^{-\frac{2\pi i}{d}} v$. It is not hard to check that α_2 extends to an automorphism of $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d$ of degree d. In this way, we can define the iterated crossed product $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ as the universal C^*-algebra generated by all sums of the form

$$F = \sum_{\ell,m=0}^{d-1} A_\ell v^\ell w^m \ , \ A_\ell \in \mathcal{U}_{nc}(d),$$

where the product and adjoint are extended from $\mathcal{U}_{nc}(d)$ to satisfy, for each $A \in \mathcal{U}_{nc}(d)$,

$$\begin{aligned} &vAv^{-1} = \alpha_1(A) \ , \ v^* = v^{-1} = v^{d-1} \ , \ vw = e^{\frac{2\pi i}{d}} wv \ , \\ &wAw^{-1} = \alpha_2(A) \ , \ w^* = w^{-1} = w^{d-1} \ . \end{aligned} \tag{5.1.3}$$

For the corresponding reduced crossed product, we recall that the generalized Pauli matrices X and Z in M_d are given by

$$Xe_j = e^{\frac{2\pi i j}{d}} e_j, \;\; Ze_j = e_{j+1}, \; \forall 0 \leq j \leq d-1.$$

Identifying $\ell^2(\mathbb{Z}_d) \simeq \mathbb{C}^d$ as Hilbert spaces, the left regular representation $\lambda : \mathbb{Z}_d \to M_d$ is given by $\lambda(v) = Z$, where v is a generator of $\mathbb{Z}_d$. If we let $\rho : \mathcal{U}_{nc}(d) \to M_d \otimes \mathcal{U}_{nc}(d)$ be defined by

$$\rho(A) = \sum_{m=0}^{d-1} E_{mm} \otimes \alpha_1^{-m}(A),$$

then by Definition 1.5.3, the reduced crossed product $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d$ is isomorphic to the C^*-subalgebra of $M_d \otimes \mathcal{U}_{nc}(d)$ generated by the range of ρ and the unitary $Z \otimes 1$, via the flip map $M_d \otimes \mathcal{U}_{nc}(d) \to \mathcal{U}_{nc}(d) \otimes M_d$. We will denote this isomorphism by γ.

For the action α_2, we define $\pi : \mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d \to M_d \otimes M_d \otimes \mathcal{U}_{nc}(d)$ by

$$\pi(B) = \sum_{\ell=0}^{d-1} E_{\ell\ell} \otimes \gamma(\alpha_2^{-\ell}(B)), \, \forall B \in \mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d.$$

Then for $A \in \mathcal{U}_{nc}(d)$, we have

$$\pi(A) = \sum_{\ell,m=0}^{d-1} E_{\ell\ell} \otimes E_{mm} \otimes \alpha_1^{-m}\alpha_2^{-\ell}(A),$$

while, if v is the generator of $\mathbb{Z}_d$ in $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d$,

$$\begin{aligned} \pi(v) &= \sum_{\ell=0}^{d-1} E_{\ell\ell} \otimes \gamma(\alpha_2^{-\ell}(v)) \\ &= \sum_{\ell=0}^{d-1} E_{\ell\ell} \otimes e^{\frac{2\pi i \ell}{d}} \gamma(v) \\ &= X \otimes Z \otimes 1. \end{aligned}$$

Using the left regular representation of $\mathbb{Z}_d$, we see that $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d \rtimes_{\alpha_2,r} \mathbb{Z}_d$ is isomorphic to the subalgebra of $M_d \otimes M_d \otimes \mathcal{U}_{nc}(d)$ generated by the range of π and the unitary $Z \otimes 1 \otimes 1$. For convenience, we will flip the two tensor factors of M_d. In this way, the iterated reduced crossed product $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d \rtimes_{\alpha_2,r} \mathbb{Z}_d$ is isomorphic to the C^*-subalgebra of $M_d \otimes M_d \otimes \mathcal{U}_{nc}(d)$ generated by the range of the map $\pi : \mathcal{U}_{nc}(d) \to M_d \otimes M_d \otimes \mathcal{U}_{nc}(d)$ given by

$$\pi(A) = \sum_{\ell,m=0}^{d-1} E_{\ell\ell} \otimes E_{mm} \otimes \alpha_1^{-\ell} \alpha_2^{-m}(A) ,$$

and by the unitaries $v = Z \otimes X \otimes 1$ and $w = 1 \otimes Z \otimes 1$. Since $\mathbb{Z}_d$ is amenable, Theorem 1.5.5 shows that the full crossed product $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ isomorphic to the reduced crossed product $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d \rtimes_{\alpha_2,r} \mathbb{Z}_d$ via the canonical quotient map.

We also define two automorphisms β_1, β_2 on $C^*(\mathbb{F}_{d^2})$ by

$$\beta_1(g_{jk}) = g_{j+1,k} \ , \ \beta_2(g_{jk}) = g_{j,k-1} \ , \ 0 \leq j, k \leq d-1 \ .$$

The iterated crossed product $C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d$ is defined in a similar manner to (5.1.3). We are now in a position to prove Theorem 5.0.6.

Theorem 5.1.5. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the automorphisms given above. Then*

$$\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \cong M_d(C^*(\mathbb{F}_{d^2})) \ \ \textit{and} \ \ C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d \cong M_d(\mathcal{U}_{nc}(d)) \ .$$

Proof. Let $\mathcal{L}_1 : \mathcal{U}_{nc}(d) \to M_d(C^*(\mathbb{F}_{d^2}))$ and $\mathcal{L}_2 : C^*(\mathbb{F}_{d^2}) \to M_d(\mathcal{U}_{nc}(d))$ be the unital $*$-homomorphisms from Lemma 5.1.4. Then $(\mathcal{L}_1, X \otimes 1)$ is a covariant representation of the C^*-dynamical system $(\mathcal{U}_{nc}(d), \alpha_1, \mathbb{Z}_d)$. By the universal property of $\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d$, this covariant representation induces a canonical $*$-homomorphism $\mathcal{L}_1' : \mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \to M_d(C^*(\mathbb{F}_{d^2}))$ such that, for $A \in \mathcal{U}_{nc}(d)$ and the generator v of $\mathbb{Z}_d$,

$$\mathcal{L}_1'(A) = \mathcal{L}_1(A) \ , \ \mathcal{L}_1'(v) = X \otimes 1 \ .$$

Moreover, $(\mathcal{L}_1', Z \otimes 1)$ is a covariant representation of $(\mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d, \alpha_2, \mathbb{Z}_d)$, so it induces a canonical $*$-homomorphism $\widetilde{\mathcal{L}}_1 : \mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \to M_d(C^*(\mathbb{F}_{d^2}))$ such that, for $A \in \mathcal{U}_{nc}(d)$ and the generators v, w of the two copies of $\mathbb{Z}_d$,

$$\widetilde{\mathcal{L}}_1(A) = \mathcal{L}_1(A) \ , \ \widetilde{\mathcal{L}}_1(v) = X \otimes 1 \ , \ \widetilde{\mathcal{L}}_1(w) = Z \otimes 1 \ .$$

Considering the form of $\mathcal{L}(u_{jk})$ for each j, k, one can see that $\widetilde{\mathcal{L}}_1$ is surjective. Indeed, this follows since $M_d = \operatorname{span}\{T_{j,k} : 0 \leq j, k \leq d-1\}$, so that any element of the form $E_{jk} \otimes g_{\ell m}$ lies in the algebra generated by $\mathcal{L}_1(\mathcal{U}_{nc}(d)) \cup (M_d \otimes \mathbb{C}1)$. Now, consider the $*$-homomorphism $\mathrm{id}_{M_d} \otimes \mathcal{L}_2 : M_d(C^*(\mathbb{F}_{d^2})) \to M_d \otimes M_d \otimes \mathcal{U}_{nc}(d)$. The range of $id_{M_d} \otimes \mathcal{L}_2$

is generated by $(id_{M_d}\otimes\mathcal{L}_2)\circ\mathcal{L}_1(\mathcal{U}_{nc}(d))$ and $M_d\otimes\mathbb{C}1\otimes\mathbb{C}1$. Note that for $0\le j,k\le d-1$,

$$\begin{aligned}(id_{M_d}\otimes\mathcal{L}_2)\circ\mathcal{L}_1(u_{jk}) &= \frac{1}{d}\sum_{\ell,m} e^{-\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m}\otimes\mathcal{L}_2(g_{\ell m})\\ &= \frac{1}{d}\sum_{\ell,m,a,b} e^{-\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m}\otimes e^{\frac{2\pi i(a-b)\ell}{d}} E_{a-m,b-m}\otimes u_{ab}\\ &= \frac{1}{d}\sum_{\ell,m,a,b} E_{j-m,k-m}\otimes e^{\frac{2\pi i((a-b)-(j-k))\ell}{d}} E_{a-m,b-m}\otimes u_{ab}.\end{aligned}$$

Fixing a,b,m, the sum over ℓ of the above expression will be non-zero if and only if the exponent is 0. The exponent is 0 when $a-b=j-k$. Thus, the sum does not change when we replace the index (a,b) with $(j+n,k+n)$, where n varies over $\{0,...,d-1\}$. Then we may write

$$(\mathrm{id}_{M_d}\otimes\mathcal{L}_2)\circ\mathcal{L}_1(u_{jk}) = \sum_{m,n} E_{j-m,k-m}\otimes E_{j+n-m,k+n-m}\otimes u_{j+n,k+n}.$$

By a similar argument, the latter quantity is also equal to

$$\frac{1}{d}\sum_{\ell,n,x,y} E_{x-n,y-n}\otimes E_{x,y}\otimes e^{\frac{2\pi i((j-k)-(x-y))\ell}{d}} u_{j+n,k+n}.$$

Therefore,

$$\begin{aligned}(\mathrm{id}_{M_d}\otimes\mathcal{L}_2)\circ\mathcal{L}_1(u_{jk}) &= \sum_{\ell,n,x,y} e^{-\frac{2\pi i(x-y)\ell}{d}} E_{x-n,y-n}\otimes E_{x,y}\otimes\alpha_1^\ell\alpha_2^n(u_{jk})\\ &= \sum_{\ell,n}\phi_{-\ell,-n}\phi^*_{-\ell,-n}\otimes\alpha_1^\ell\alpha_2^n(u_{jk})\ .\end{aligned}$$

Let $V\in M_d\otimes M_d$ be the unitary given by $V(e_j\otimes e_k)=e^{-\frac{2\pi ijk}{d}}\phi_{jk}$ for each $0\le j,k\le d-1$. Then for all $A\in\mathcal{U}_{nc}(d)$,

$$(V^*\otimes 1)(id_{M_d}\otimes\mathcal{L}_2)\circ\mathcal{L}_1(A)(V\otimes 1) = \sum_{j,k=0}^{d-1} E_{jj}\otimes E_{kk}\otimes\alpha_1^{-j}\alpha_2^{-k}(A)\ , \qquad (5.1.4)$$

while, for each $0 \leq j, k \leq d-1$,

$$\begin{aligned}
V^*(X \otimes \mathrm{id}_{M_d})V(e_j \otimes e_k) &= e^{-\frac{2\pi ijk}{d}} V^*(X \otimes 1)\phi_{jk} \\
&= e^{-\frac{2\pi ijk}{d}} V^*\phi_{j+1,k} \\
&= e^{\frac{2\pi ik}{d}} (e_{j+1} \otimes e_k) \\
&= Ze_j \otimes Xe_k.
\end{aligned} \tag{5.1.5}$$

Moreover,

$$\begin{aligned}
V^*(Z \otimes \mathrm{id}_{M_d})V(e_j \otimes e_k) &= e^{-\frac{2\pi ijk}{d}} V^*(Z \otimes \mathrm{id}_{M_d})\phi_{jk} \\
&= e^{-\frac{2\pi ijk}{d}} V^*(ZX^jZ^k \otimes \mathrm{id}_{M_d})(e_j \otimes e_k) \\
&= e^{-\frac{2\pi ij(k+1)}{d}} V^*(X^jZ^{k+1} \otimes \mathrm{id}_{M_d})(e_j \otimes e_k) \\
&= e^{-\frac{2\pi ij(k+1)}{d}} V^*\phi_{j,k+1} \\
&= e_j \otimes e_{k+1} = (\mathrm{id}_{M_d} \otimes Z)(e_j \otimes e_k).
\end{aligned} \tag{5.1.6}$$

Combining equations (5.1.5) and (5.1.6), we obtain

$$V^*(X \otimes 1)V = Z \otimes X \ , \ V^*(Z \otimes 1)V = 1 \otimes Z. \tag{5.1.7}$$

In particular, $(V^* \otimes 1)(M_d \otimes \mathcal{L}_2(C^*(\mathbb{F}_{d^2})))(V \otimes 1) = \mathcal{U}_{nc}(d) \rtimes_{\alpha_1,r} \mathbb{Z}_d \rtimes_{\alpha_2,r} \mathbb{Z}_d$. It follows that the map $(V^* \otimes 1)[(id_{M_d} \otimes \mathcal{L}_2) \circ \widetilde{\mathcal{L}_1}(\cdot)](V \otimes 1)$ is the canonical quotient map from the full crossed product to the reduced crossed product, and must be a $*$-isomorphism. Therefore, $\widetilde{\mathcal{L}_1}$ is injective, so that $\widetilde{\mathcal{L}_1}$ is also a $*$-isomorphism.

The argument for the second isomorphism is similar. Since $\mathcal{L}_2(g_{\ell m}) = (T_{\ell,-m} \otimes 1)U(T_{\ell,-m} \otimes 1)^*$ for all ℓ, m, using covariant representations, we obtain the surjective $*$-homomorphism

$$\begin{aligned}
&\widetilde{\mathcal{L}}_2 : C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d \to M_d(\mathcal{U}_{nc}(d)) \ , \\
&\widetilde{\mathcal{L}}_2(B) = \mathcal{L}_2(B) \ , \ \widetilde{\mathcal{L}}_2(v) = X \otimes 1 \ , \ \widetilde{\mathcal{L}}_2(w) = Z \otimes 1 \ .
\end{aligned}$$

Note that for each $0 \leq \ell, m \leq d-1$,

$$\begin{aligned}(id_{M_d} \otimes \mathcal{L}_1) \circ \mathcal{L}_2(g_{\ell m}) &= \frac{1}{d} \sum_{j,k} e^{\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes \mathcal{L}_1(u_{jk}) \\ &= \frac{1}{d} \sum_{j,k,a,b} e^{\frac{2\pi i(j-k)\ell}{d}} E_{j-m,k-m} \otimes e^{-\frac{2\pi i(j-k)a}{d}} E_{j-b,k-b} \otimes g_{ab} \\ &= \frac{1}{d} \sum_{j,k,a,b} e^{\frac{2\pi i(j-k)(\ell-a)}{d}} E_{j-m,k-m} \otimes E_{j-b,k-b} \otimes \beta_1^{a-\ell} \beta_2^{m-b}(g_{\ell m}) \\ &= \sum_{a,b} \phi_{\ell-a,-m+b} \phi^*_{\ell-a,-m+b} \otimes \beta_1^{a-\ell} \beta_2^{m-b}(g_{\ell m}) \, . \\ &= \sum_{a,b} \phi_{a,b} \phi^*_{a,b} \otimes \beta_1^{-a} \beta_2^{-b}(g_{\ell m}) \, . \end{aligned} \tag{5.1.8}$$

Conjugating by the same unitary $V \otimes 1$ as in (5.2.13), we obtain the canonical quotient map from the full crossed product to the reduced crossed product, which is a $*$-isomorphism. Hence, $\widetilde{\mathcal{L}}_2$ is a $*$-isomorphism. □

Corollary 5.1.6. *There exist unital completely positive maps $\mathcal{N}_1 : M_d(C^*(\mathbb{F}_{d^2})) \to \mathcal{U}_{nc}(d)$ and $\mathcal{N}_2 : M_d(\mathcal{U}_{nc}(d)) \to C^*(\mathbb{F}_{d^2})$ such that $\mathcal{N}_1 \circ \mathcal{L}_1 = id_{\mathcal{U}_{nc}(d)}$ and $\mathcal{N}_2 \circ \mathcal{L}_2 = id_{C^*(\mathbb{F}_{d^2})}$. As a consequence, $\mathcal{L}_1$ and $\mathcal{L}_2$ are injective $*$-homomorphisms.*

Proof. By Proposition 1.5.4, there are natural conditional expectations $\mathcal{E}_1 : \mathcal{U}_{nc}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \to \mathcal{U}_{nc}(d)$ and $\mathcal{E}_2 : C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d \to C^*(\mathbb{F}_{d^2})$ given by

$$\mathcal{E}_1 \left(\sum_{\ell,m=0}^{d-1} A_{\ell m} v^\ell w^m \right) = A_{00} \text{ and } \mathcal{E}_2 \left(\sum_{\ell,m=0}^{d-1} B_{\ell m} v^\ell w^m \right) = B_{00} \, .$$

Define

$$\mathcal{N}_1 = \mathcal{E}_1 \circ \widetilde{\mathcal{L}}_1^{-1} \text{ and } \mathcal{N}_2 = \mathcal{E}_2 \circ \widetilde{\mathcal{L}}_2^{-1} \, ,$$

where $\widetilde{\mathcal{L}}_1$ and $\widetilde{\mathcal{L}}_2$ are the isomorphisms from Theorem 5.1.5. It readily follows that $\mathcal{N}_1 \circ \mathcal{L}_1 = \mathrm{id}_{\mathcal{U}_{nc}(d)}$ and $\mathcal{N}_2 \circ \mathcal{L}_2 = \mathrm{id}_{C^*(\mathbb{F}_{d^2})}$. □

Isomorphisms analogous to those in Theorem 5.1.5 can be obtained for the reduced C^*-algebras associated to $C^*(\mathbb{F}_{d^2})$ and $\mathcal{U}_{nc}(d)$. Let $C(\mathbb{T})$ be the C^*-algebra of continuous functions on the unit circle. K. McClanahan related $\mathcal{U}_{nc}(d)$ to $C(\mathbb{T})$ by using the free product of M_d and $C(\mathbb{T})$, amalgamated over the identity.

Proposition 5.1.7. (McClanahan, [44, Proposition 2.2]) *For $d \geq 2$, $M_d(\mathcal{U}_{nc}(d))$ is isomorphic to $M_d * C(\mathbb{T})$ via the map*

$$u_{jk} \mapsto \sum_{\ell=0}^{d-1} E_{\ell j} u E_{k\ell}, \;\; E_{jk} \mapsto E_{jk},$$

where u is the unitary $u(z) = z$ for all $z \in \mathbb{T}$. Moreover, the image of $\mathcal{U}_{nc}(d) = I_d \otimes \mathcal{U}_{nc}(d)$ under this map is the relative commutant M_d^c in $M_d * C(\mathbb{T})$.

Motivated by Proposition 5.1.7, McClanahan defined the reduced Brown algebra, denoted $\mathcal{U}_{nc}^{red}(d)$, as the relative commutant M_d^c in the reduced free product $M_d *_{red} C(\mathbb{T})$, where the reduced free product is taken with respect to the unique (normalized) trace tr on M_d and the canonical trace τ on $C(\mathbb{T})$ given by

$$\mathrm{tr}([a_{jk}]) = \frac{1}{d}\sum_j a_{jj} \;\text{ and } \tau(f) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt\;.$$

We recall that the reduced group C^*-algebra of $C^*_{red}(\mathbb{F}_d)$ is the C^*-algebra generated by the range of the left-regular representation

$$\lambda : \mathbb{F}_d \to \mathcal{B}(\ell_2(\mathbb{F}_d))\;,\; \text{where }\; \lambda(g)\delta_h = \delta_{gh}\;,\; \forall\, g, h \in \mathbb{F}_d.$$

The analogue of Theorem 5.1.5 for the reduced case relies on the following lemma.

Lemma 5.1.8. *Let $(\mathcal{A}, \phi)$ and $(\mathcal{C}, \psi)$ be two unital C^*-algebras equipped with states ϕ and ψ, respectively. Denote by π_ϕ (respectively π_ψ) the GNS-representation of ϕ (respectively ψ). Suppose that $\alpha : \mathcal{A} \to \mathcal{C}$ is a $*$-isomorphism satisfying $\phi = \psi \circ \alpha$. Then there exists a $*$-isomorphism $\alpha_\pi : \pi_\phi(\mathcal{A}) \to \pi_\psi(\mathcal{C})$ such that $\alpha_\pi \circ \pi_\phi = \pi_\psi \circ \alpha_\pi$.*

Proof. Let $L_2(\mathcal{A}, \phi)$, $L_2(\mathcal{C}, \psi)$ be the Hilbert spaces of the GNS construction for ϕ and ψ, respectively. For any $a_1, a_2 \in \mathcal{A}$, we have

$$\psi(\alpha(a_1)^*\alpha(a_2)) = \psi(\alpha(a_1^* a_2)) = \phi(a_1^* a_2)\;.$$

Thus, α induces a unitary operator $V_\alpha : L_2(\mathcal{A}, \phi) \to L_2(\mathcal{C}, \psi)$ such that $V_\alpha(\widehat{a}) = \widehat{\alpha(a)}$ for each $a \in \mathcal{A}$, where $\widehat{a}$ is the image of a in $L^2(\mathcal{A}, \phi)$ and $\widehat{\alpha(a)}$ is the image of $\alpha(a)$ in $L^2(\mathcal{C}, \psi)$. Consider

$$\alpha_\pi(\cdot) = V_\alpha(\cdot)V_\alpha^*$$

as a $*$-isomorphism from $\mathcal{B}(L_2(\mathcal{A},\phi))$ onto $\mathcal{B}(L_2(\mathcal{C},\psi))$. Note that for every $a, b \in \mathcal{A}$, we have

$$\alpha_\pi(\pi_\phi(a))\widehat{\alpha(b)} = V_\alpha \pi_\phi(a) V_\alpha^* \widehat{\alpha(b)} = V_\alpha \pi_\phi(a)\widehat{b} = V_\alpha(\widehat{ab}) = \widehat{\alpha(a)\alpha(b)} = \pi_\psi(\alpha(a))\widehat{\alpha(b)} \,.$$

Since $\alpha : \mathcal{A} \to \mathcal{C}$ is a $*$-isomorphism, $\alpha_\pi : \pi_\phi(\mathcal{A}) \to \pi_\psi(\mathcal{C})$ must be a $*$-isomorphism satisfying $\alpha_\pi \circ \pi_\phi = \pi_\psi \circ \alpha_\pi$, completing the proof. $\square$

Note that the reduced free product $M_d *_{red} C(\mathbb{T})$ is isomorphic to the range of the GNS representation of the full free product $M_d * C(\mathbb{T})$ with respect to free product trace $tr * \tau$. Similarly, the reduced group C^*-algebra $C^*_{red}(\mathbb{F}_{d^2})$ is isomorphic to the range of the GNS representation of $C^*(\mathbb{F}_{d^2})$ with respect to the canonical trace $\omega : C^*(\mathbb{F}_{d^2}) \to \mathbb{C}$ given on finite sums by

$$\omega\left(\sum_{\gamma \in \mathbb{F}_{d^2}} a_\gamma \gamma\right) = a_1.$$

A calculation shows that, under the isomorphism of Proposition 5.1.7, the fundamental unitary $U = \sum_{j,k=0}^{d-1} E_{jk} \otimes u_{jk}$ in $M_d(\mathcal{U}_{nc}(d))$ is sent to u in $C(\mathbb{T})$. We recall that the set $\{T_{j,k} : 0 \leq j, k \leq d-1\}$ is an orthonormal basis for M_d with respect to the Hilbert-Schmidt norm, where $T_{j,k} = X^j Z^k$ and X, Z are the generalized Pauli matrices in M_d. Since

$$\ker(\mathrm{tr}) = \mathrm{span}\ \{T_{j,k} : 0 \leq j, k \leq d-1,\ (j,k) \neq (0,0)\}$$

and

$$\ker(\tau) = \overline{\mathrm{span}\{u^n : n \in \mathbb{Z} \setminus \{0\}\}}^{\|\cdot\|},$$

applying Proposition 1.6.1, the free product trace $\mathrm{tr} * \tau$ is the unique state on $M_d(\mathcal{U}_{nc}(d))$ satisfying

- $(\mathrm{tr} * \tau)(T_{j,k} \otimes 1_{\mathcal{U}_{nc}(d)}) = \mathrm{tr}(T_{j,k})$ for all $0 \leq j, k \leq d-1$,
- $(\mathrm{tr} * \tau)\,(U^m) = \tau(u^m)$ for all $m \in \mathbb{Z}$, and
- $(\mathrm{tr} * \tau)(C) = 0$ whenever C is a word with alternating letters from $\{T_{j,k} : (j,k) \neq (0,0)\}$ and $\{U^n : n \in \mathbb{Z} \setminus \{0\}\}$.

The isomorphism $\mathrm{id}_d \otimes \alpha_1$ fixes the copy of M_d in $M_d(\mathcal{U}_{nc}(d))$, and sends the fundamental unitary U to $XUX^* = T_{1,0} U T_{1,0}^*$. It is not hard to see, then, that $\mathrm{id}_d \otimes \alpha_1$ preserves the free product trace $\mathrm{tr} * \tau$. Similarly, $\mathrm{id}_d \otimes \alpha_2(U) = ZUZ^* = T_{0,1} U T_{0,1}^*$, so that $\mathrm{id}_d \otimes \alpha_2$ preserves the trace. By Lemma 5.1.8, the actions α_1, α_2 on $\mathcal{U}_{nc}(d)$ induce reduced versions of the actions. Similarly, the actions β_1, β_2 on $C^*(\mathbb{F}_{d^2})$ induce reduced versions of the actions because they preserve the trace ω. For simplicity, the reduced versions of the actions on $\mathcal{U}_{nc}(d)$ and $C^*(\mathbb{F}_{d^2})$ will also be denoted by $\alpha_1, \alpha_2, \beta_1$ and β_2.

Corollary 5.1.9. *Let α_1 and α_2 the the actions on $\mathcal{U}_{nc}^{red}(d)$ induced by the actions on $\mathcal{U}_{nc}(d)$, and let β_1, β_2 be the actions on $C^*_{red}(\mathbb{F}_{d^2})$ induced by the actions on $C^*(\mathbb{F}_{d^2})$. Then*

$$\mathcal{U}_{nc}^{red}(d) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \cong M_d(C^*_{red}(\mathbb{F}_{d^2})) \ \text{ and } \ C^*_{red}(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d \cong M_d(\mathcal{U}_{nc}^{red}(d)) \ .$$

Proof. We begin by proving the second isomorphism. Because β_1 and β_2 preserve the trace ω, the (reduced) crossed product $C^*_{red}(\mathbb{F}_{d^2}) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ is the GNS representation of $C^*(\mathbb{F}_{d^2}) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ with respect to the natural extension of ω to $C^*(\mathbb{F}_{d^2}) \rtimes_{\beta_1} \mathbb{Z}_d \rtimes_{\beta_2} \mathbb{Z}_d$ given by

$$\widehat{\omega}\left(\sum_{l,m=0}^{d-1} A_{lm} v^l w^m\right) = \omega(A_{00}) \ .$$

Thus, it is sufficient to show that the isomorphism $\widetilde{\mathcal{L}}_2$ from Theorem 5.1.5 is trace-preserving. That is to say, it is sufficient to show that $(tr * \tau) \circ \widetilde{\mathcal{L}}_2 = \widehat{\omega}$. Consider a reduced word $g = g_{j_1,k_1}^{\epsilon_1} g_{j_2,k_2}^{\epsilon_2} \cdots g_{j_n,k_n}^{\epsilon_n}$ in $\mathbb{F}_{d^2}$, where $(j_a, k_a) \neq (j_{a+1}, k_{a+1})$ for $1 \leq a \leq n-1$, and $\epsilon_1, \cdots, \epsilon_n \in \mathbb{Z} \setminus \{0\}$. Recall that $T_{j,k} = X^j Z^k$. Let $U = \sum_{j,k} E_{jk} \otimes u_{jk}$ be the fundamental unitary of $M_d(\mathcal{U}_{nc}(d))$, and let

$$U_{j,k} := (T_{j,-k} \otimes 1) U (T_{j,-k} \otimes 1)^* = \widetilde{\mathcal{L}}_2(g_{j,k}) \ .$$

We observe that for $0 \leq a_1, a_2, b_1, b_2 \leq d-1$, we have

$$T^*_{a_1,b_1} T_{a_2,b_2} = Z^{-b_1} X^{a_2-a_1} Z^{b_2} = \exp\left(\frac{2\pi i b_1 (a_2 - a_1)}{d}\right) T_{a_2-a_1, b_2-b_1}.$$

Then for $0 \leq j_{n+1}, k_{n+1} \leq d$,

$$\widetilde{\mathcal{L}}_2(g v^{j_{n+1}} w^{k_{n+1}}) = U_{j_1,k_1}^{\epsilon_1} U_{j_2,k_2}^{\epsilon_2} \cdots U_{j_n,k_n}^{\epsilon_n} (T_{j_{n+1},k_{n+1}} \otimes 1)$$

Using the fact that $U_{j_\ell,k_\ell}^{\varepsilon_\ell} = (T_{j_\ell,-k_\ell} \otimes 1) U^{\varepsilon_\ell} (T_{j_\ell,-k_\ell} \otimes 1)^*$, we find that

$$\widetilde{\mathcal{L}}_2(g v^{j_{n+1}} w^{k_{n+1}}) = \lambda \cdot (T_{j_1,-k_1} \otimes 1) U^{\epsilon_1} (T_{j_2-j_1,k_1-k_2} \otimes 1) \cdots U^{\epsilon_n} (T_{j_{n+1}-j_n, k_n+k_{n+1}} \otimes 1) \ ,$$

for some constant $\lambda \in \mathbb{C}$. If $j_{n+1} - j_n \neq 0$ or $k_{n+1} + k_n \neq 0$ modulo d, then by definition of the free product trace $tr * \tau$, the element $\widetilde{\mathcal{L}}_2(gv^{j_{n+1}}w^{k_{n+1}})$ has trace zero. If $j_{n+1} - j_n = 0 = k_{n+1} + k_n$ modulo d, then $\widetilde{\mathcal{L}}_2(gv^{j_{n+1}}w^{k_{n+1}})$ is an alternating product of elements from $\ker(\mathrm{tr})$ and $\ker(\tau)$, so that $\mathrm{tr} * \tau(\widetilde{\mathcal{L}}_2(gv^{j_{n+1}}w^{k_{n+1}})) = 0$. When $g = 1$,

$$\widetilde{\mathcal{L}}_2(v^j w^k) = T_{j,k} \otimes 1,$$

and this element has trace 1 if $(j,k) = (0,0)$ and trace 0 otherwise. It follows that $(tr * \tau) \circ \widetilde{\mathcal{L}}_2 = \omega$, so that the second isomorphism follows from Lemma 5.1.8. For the first isomorphism, we start with a matrix version of the full algebras:

$$M_d \otimes M_d(C^*(\mathbb{F}_{d^2})) \cong M_d(\mathcal{U}_{nc}(d)) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d \cong M_d *_{\mathbb{C}} C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d\ ,$$

where the actions α_1, α_2 are extended to $M_d(\mathcal{U}_{nc}(d))$ by fixing the copy of M_d. We want to show that the extension of free product trace $\widehat{tr * \tau}$ on $M_d * C(\mathbb{T}) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ coincides with the product trace $tr \otimes tr \otimes \omega$ on $M_d \otimes M_d(C^*(\mathbb{F}_{d^2}))$. Now, $M_d(\mathcal{U}_{nc}(d)) \rtimes_{\alpha_1} \mathbb{Z}_d \rtimes_{\alpha_2} \mathbb{Z}_d$ is spanned by elements of the form

$$U_{j_1k_1}^{\epsilon_1} U_{j_2k_2}^{\epsilon_2} \cdots U_{j_nk_n}^{\epsilon_n} (X^{j_{n+1}} Z^{k_{n+1}} \otimes 1) v^{j_{n+2}} w^{k_{n+2}}\ , \tag{5.1.9}$$

for $(j_1,k_1) \neq (j_2,k_2) \neq \cdots \neq (j_n,k_n)$ and nonzero $\epsilon_1, \cdots, \epsilon_n \in \mathbb{Z}$. As before, the trace $\widehat{tr * \tau}$ of (5.1.9) is 1 if the word equals the identity (which happens when $n = j_{n+1} = j_{n+2} = k_{n+1} = k_{n+2} = 0$), and 0 otherwise. If $0 \leq j,k \leq d-1$, then we can apply equation (5.1.8) to find that

$$\begin{aligned}
(\mathrm{id}_{M_d} \otimes \widetilde{\mathcal{L}}_1)(U_{jk}) &= (\mathrm{id}_{M_d} \otimes \widetilde{\mathcal{L}}_1)(\widetilde{\mathcal{L}}_2(g_{jk})) \\
&= \sum_{\ell,m} \phi_{\ell,m}\phi_{\ell,m}^* \otimes \beta_1^{-\ell}\beta_2^{-m}(g_{jk}) \\
&= \sum_{\ell,m} \phi_{\ell,m}\phi_{\ell,m}^* \otimes g_{j-\ell,m+k}.
\end{aligned}$$

Thus, the isomorphism $id_{M_d} \otimes \widetilde{\mathcal{L}_1}$ sends an element of the form in (5.1.9) to

$$\sum_{\ell,m} e^{2\pi i j_{n+1}(k_{n+1}-k_{n+2})} \phi_{\ell,m}\phi_{\ell,m}^* (T_{j_{n+1},k_{n+1}} \otimes T_{j_{n+2},k_{n+2}}) \otimes g_{j_1-\ell,m+k_1}^{\epsilon_1} \cdots g_{j_n-\ell,m+k_n}^{\epsilon_n}.$$

An element of this form is sent to 0 by $tr \otimes tr \otimes \omega$, because the word $g_{j_1-l,m+k_1}^{\epsilon_1} \cdots g_{j_n-l,m+k_n}^{\epsilon_n}$ is reduced and non-trivial. When $n = 0$,

$$id_{M_d} \otimes \widetilde{\mathcal{L}}_1((T_{j_1,k_1} \otimes 1)v^{j_2}w^{k_2}) = T_{j_1,k_1} \otimes T_{j_2,k_2} \otimes 1,$$

and this element is sent by $\mathrm{tr}\otimes\mathrm{tr}\otimes\omega$ to 1 if $(j_1,k_1)=(j_2,k_2)=(0,0)$ and 0 otherwise. Hence, the trace is preserved. By Lemma 5.1.8, we have that

$$M_d\otimes M_d(C^*_{red}(\mathbb{F}_{d^2}))\cong M_d(\mathcal{U}_{nc}^{red}(d))\rtimes_{\alpha_1}\mathbb{Z}_d\rtimes_{\alpha_2}\mathbb{Z}_d\,.$$

Note that this isomorphism maps $M_d\otimes\mathbb{C}1\otimes\mathbb{C}1$ to $M_d\otimes\mathbb{C}1$. Therefore, the first isomorphism in the theorem statement follows from taking the relative commutant. $\square$

For the corresponding von Neumann algebras associated with the reduced algebras, we have the following remark.

Remark 5.1.10. Let $L(\mathbb{F}_{d^2})$ be the free group factor corresopnding to $\mathbb{F}_{d^2}$. That is, $L(\mathbb{F}_{d^2})$ is the weak*-closure of $C^*_{red}(\mathbb{F}_{d^2})$. K. Dykema [18, Theorem 3.4] proved the following formula:

$$M_d *_{red} L(\mathbb{F}_k)\cong L(\mathbb{F}_{d^2k})\otimes M_d\,,\ \forall\, 1\le k\le\infty, d\ge 2\,.$$

When $k=1$, $M_d(\overline{\mathcal{U}_{nc}^{red}(d)}^{w^*})\cong M_d *_{red} L(\mathbb{Z})\cong M_d(L(\mathbb{F}_{d^2}))$, which implies that the von Neumann algebra $\overline{\mathcal{U}_{nc}^{red}(d)}^{w^*}$ is isomorphic to $L(\mathbb{F}_{d^2})$. Note that the actions $\alpha_1,\alpha_2,\beta_1,\beta_2$ are all trace preserving. Then the two isomorphisms of Corollary 5.1.9 merge when taking weak*-closure. That is to say, we have the isomorphisms

$$M_d(L(\mathbb{F}_{d^2}))\cong L(\mathbb{F}_{d^2})\rtimes_{\alpha_1}\mathbb{Z}_d\rtimes_{\alpha_2}\mathbb{Z}_d\cong L(\mathbb{F}_{d^2})\rtimes_{\beta_1}\mathbb{Z}_d\rtimes_{\beta_2}\mathbb{Z}_d\,.$$

5.2 Matrix-valued quantum correlation sets

In this section, we will use the embezzlement state on $\mathcal{U}_{nc}(2)\otimes_{\min}\mathcal{U}_{nc}(2)$ from Chapter 3 to separate matrix versions of C_{qs} and C_{qa}. Along the way, we will also exhibit separations in the matrix-valued versions of what are referred to as the **free correlation sets**. The free correlation sets are an analogue of quantum correlations for $C^*(\mathbb{F}_d)$. Motivated by the hierarchy of quantum correlation sets, we define $F_q^{(n)}(d)$ to be the set of all correlations in $(M_n)^{d^2}$ of the form

$$\left((\langle(u_j\otimes v_k)\eta_q,\eta_p\rangle)_{p,q=1}^n\right)_{j,k=0}^{d-1},$$

where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional Hilbert spaces, $\{u_j\}_{j=0}^{d-1}$ are unitaries on $\mathcal{H}_A$, $\{v_k\}_{k=0}^{d-1}$ are unitaries on $\mathcal{H}_B$, and $\{\eta_1,...,\eta_n\}$ is a collection of orthonormal vectors in $\mathcal{H}_A\otimes\mathcal{H}_B$.

We define $F_{qs}^{(n)}(d)$ in the same manner, dropping the requirement that $\mathcal{H}_A$ and $\mathcal{H}_B$ be finite-dimensional. We define $F_{qc}^{(n)}(d)$ as the set of all correlations of the form

$$\left((\langle u_j v_k \eta_q, \eta_p \rangle)_{p,q=1}^n\right)_{j,k=0}^{d-1},$$

where $\mathcal{H}$ is a Hilbert space, $u_0, ..., u_{d-1}, v_0, ..., v_{d-1}$ are unitaries on $\mathcal{H}$ with $[u_j, v_k] = 0$ for all j, k, and $\{\eta_1, ..., \eta_n\}$ is an orthonormal set in $\mathcal{H}$. We will set $F_{qa}^{(n)}(d) = \overline{F_{qs}^{(n)}(d)}$. The arguments in Section 1.7 translate to the free correlation sets, since the C^*-algebra $C^*(\mathbb{F}_d)$ is RFD [9]. In particular, it is not hard to see that

$$F_{qa}^{(n)}(d) = \{(\Psi(g_j \otimes g_k))_{j,k=0}^{d-1} : \Psi \in \mathrm{UCP}(C^*(\mathbb{F}_d) \otimes_{\min} C^*(\mathbb{F}_d), M_n)\}$$

and

$$F_{qc}^{(n)}(d) = \{(\Psi(g_j \otimes g_k))_{j,k=0}^{d-1} : \Psi \in \mathrm{UCP}(C^*(\mathbb{F}_d) \otimes_{\max} C^*(\mathbb{F}_d), M_n)\}.$$

One can also show that $F_{qa}^{(n)}(d)$ is the closure of $F_{qs}^{(n)}(d)$.

Tsirelson's problem has an equivalent version in terms of the free correlation sets; namely, it is equivalent to determining whether $F_{qa}(d) = F_{qc}(d)$ for every $d \geq 2$ (see [47, Theorem 29]). These sets are closely related to the unitary correlation sets from Chapter 3.

We will briefly explain the motivation for the proof of Theorem 5.0.5. Using Theorem 3.3.3, there exists a state ψ on $\mathcal{U}_{nc}(2) \otimes_{min} \mathcal{U}_{nc}(2)$ such that

$$\psi(u_{j0} \otimes u_{k0}) = \frac{1}{\sqrt{2}} \delta_{jk}\ ,\ 0 \leq j, k \leq d-1, \tag{5.2.1}$$

and any state satisfying (5.2.1) is not spatial. This state corresponds to the embezzlement of entanglement introduced by van Dam and Hayden [64]. We translate this non-spatial correlation from $\mathcal{U}_{nc}(2) \otimes_{min} \mathcal{U}_{nc}(2)$ to a non-spatial M_2-valued UCP map on $C^*(\mathbb{F}_4) \otimes_{min} C^*(\mathbb{F}_4)$ by the isomorphisms obtained in Theorem 5.1.5. This non-spatial M_2-valued map on $C^*(\mathbb{F}_4) \otimes_{min} C^*(\mathbb{F}_4)$ leads to a matrix-valued free correlation that is in $F_{qa}^{(2)}(4)$ but not in the dense subset $F_{qs}^{(2)}(4)$. We then further translate the non-spatial correlation in $F_{qa}^{(2)}(4)$ to $C_{qa}^{(n)}(d, m)$, using the fact that the free groups can be embedded as subgroups into free products of cyclic groups $*_m \mathbb{Z}_d$ for any $d, m \geq 2$ with $(d, m) \neq (2, 2)$.

Let $\mathcal{L}_1 : \mathcal{U}_{nc}(2) \to M_2(C^*(\mathbb{F}_4))$ and $\mathcal{L}_2 : C^*(\mathbb{F}_4) \to M_2(\mathcal{U}_{nc}(2))$ be the embeddings given in Lemma 5.1.4. For notational convenience, we will let $\mathcal{L}_2^{(2)} = \mathrm{id}_{M_2} \otimes \mathcal{L}_2 : M_2(C^*(\mathbb{F}_4)) \to$

$M_4(\mathcal{U}_{nc}(2))$. Using the form of $(\mathrm{id}_{M_2} \otimes \mathcal{L}_2) \circ \mathcal{L}_1$ on the generators of $\mathcal{U}_{nc}(2)$, we obtain

$$(\mathrm{id}_{M_2} \otimes \mathcal{L}_2) \circ \mathcal{L}_1(u_{00}) = \begin{pmatrix} u_{00} & 0 & 0 & 0 \\ 0 & u_{11} & 0 & 0 \\ 0 & 0 & u_{11} & 0 \\ 0 & 0 & 0 & u_{00} \end{pmatrix}$$

$$(\mathrm{id}_{M_2} \otimes \mathcal{L}_2) \circ \mathcal{L}_1(u_{10}) = \begin{pmatrix} 0 & 0 & 0 & u_{10} \\ 0 & 0 & u_{01} & 0 \\ 0 & u_{01} & 0 & 0 \\ u_{10} & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathrm{id}_{M_2} \otimes \mathcal{L}_2) \circ \mathcal{L}_1(u_{01}) = \begin{pmatrix} 0 & 0 & 0 & u_{01} \\ 0 & 0 & u_{10} & 0 \\ 0 & u_{10} & 0 & 0 \\ u_{01} & 0 & 0 & 0 \end{pmatrix}$$

$$(\mathrm{id}_{M_2} \otimes \mathcal{L}_2) \circ \mathcal{L}_1(u_{11}) = \begin{pmatrix} u_{11} & 0 & 0 & 0 \\ 0 & u_{00} & 0 & 0 \\ 0 & 0 & u_{00} & 0 \\ 0 & 0 & 0 & u_{11} \end{pmatrix}$$

Consider the density matrix in $M_4 \otimes M_4$ given by

$$\rho = \frac{1}{2}(E_{00} \otimes E_{00} + E_{30} + E_{30} + E_{03} \otimes E_{03} + E_{33} \otimes E_{33}),$$

Then we can factor the embezzlement state ψ on $\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2)$ as

$$\psi = (\rho \otimes \psi) \circ (\mathcal{L}_2^{(2)} \otimes \mathcal{L}_2^{(2)}) \circ (\mathcal{L}_1 \otimes \mathcal{L}_1)\,.$$

Here the choice of ρ is not unique; we have simply chosen one of minimal rank. This factorization yields a state $\widetilde{\psi}$ on $M_2(C^*(\mathbb{F}_4)) \otimes_{min} M_2(C^*(\mathbb{F}_4))$ given by

$$\widetilde{\psi} = (\rho \otimes \psi) \circ \left(\mathcal{L}_2^{(2)} \otimes \mathcal{L}_2^{(2)}\right).$$

In other words, the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2) \arrow[r, "\mathcal{L}_1 \otimes \mathcal{L}_1"] \arrow[dd, "\psi"'] & M_2(C^*(\mathbb{F}_4)) \otimes_{\min} M_2(C^*(\mathbb{F}_4)) \arrow[d, "\mathcal{L}_2^{(2)} \otimes \mathcal{L}_2^{(2)}"] \arrow[ddl, "\widetilde{\psi}"'] \\
 & M_4(\mathcal{U}_{nc}(2)) \otimes_{\min} M_4(\mathcal{U}_{nc}(2)) \arrow[d, "\simeq"] \\
\mathbb{C} & M_4 \otimes M_4 \otimes (\mathcal{U}_{nc}(2) \otimes_{\min} \mathcal{U}_{nc}(2)) \arrow[l, "\rho \otimes \psi"]
\end{tikzcd}
$$

Since $\psi = \widetilde{\psi} \circ (\mathcal{L}_1 \otimes \mathcal{L}_1)$, the state $\widetilde{\psi}$ is not spatial, otherwise ψ would be spatial. The state $\widetilde{\psi}$ can be transformed into a ucp map $\Psi : C^*(\mathbb{F}_4) \otimes_{min} C^*(\mathbb{F}_4) \to M_4$ by Proposition 1.7.8. On the generators, the only non-zero entries are the $(0,0)$, $(0,3)$, $(3,0)$ and $(3,3)$ entries. Hence, applying a compression, we may assume that Ψ has range in M_2. On the generators, Ψ is given as follows:

$$
\begin{aligned}
&\Psi(g_{00} \otimes g_{00}) = \Psi(g_{10} \otimes g_{10}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \Psi(g_{10} \otimes g_{00}) = \Psi(g_{00} \otimes g_{10}) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}, \\
&\Psi(g_{01} \otimes g_{01}) = \Psi(g_{11} \otimes g_{11}) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \Psi(g_{01} \otimes g_{11}) = \Psi(g_{11} \otimes g_{01}) = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix}, \\
&\Psi(g_{jk} \otimes g_{lm}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ if } k \neq m \,.
\end{aligned} \tag{5.2.2}
$$

Note that Ψ cannot be spatial, otherwise the state $\widetilde{\psi}$ would be spatial. While this fact is implied by Theorem 5.2.1, a careful examination of the Schmidt decompositions involved with the map Ψ will allow us to make simplifications to the map Ψ. First, we prove the following.

Theorem 5.2.1. *Let $g_{00}, g_{10}, g_{01}, g_{11}$ be the generators of $C^*(\mathbb{F}_4)$. There do not exist $*$-homomorphisms $\pi_1, \pi_2 : C^*(\mathbb{F}_4) \to B(\mathcal{H})$ and orthonormal vectors $h_0, h_1 \in \mathcal{H} \otimes \mathcal{H}$ satisfying, for $0 \leq j, k \leq 1$,*

$$
\langle (\pi_1(g_{j0}) \otimes \pi_2(g_{k0}))h_0, h_0 \rangle = \frac{1}{\sqrt{2}}\,, \qquad \langle (\pi_1(g_{j0}) \otimes \pi_2(g_{k0}))h_0, h_1 \rangle = \frac{(-1)^{j-k}}{\sqrt{2}}\,, \tag{5.2.3}
$$

$$
\langle (\pi_1(g_{j1}) \otimes \pi_2(g_{k1}))h_1, h_1 \rangle = \frac{1}{\sqrt{2}}\,, \qquad \langle (\pi_1(g_{j1}) \otimes \pi_2(g_{k1}))h_1, h_0 \rangle = \frac{(-1)^{j-k}}{\sqrt{2}}\,. \tag{5.2.4}
$$

Proof. Suppose we have a setting described by (5.2.3) and (5.2.4). By summing up the corresponding equations in (5.2.3) and (5.2.4), we have

$$\langle(\pi_1(g_{00}) \otimes \pi_2(g_{00}))h_0, h_0 + h_1\rangle = \langle(\pi_1(g_{10}) \otimes \pi_2(g_{10}))h_0, h_0 + h_1\rangle = \sqrt{2}\,, \tag{5.2.5}$$

$$\langle(\pi_1(g_{10}) \otimes \pi_2(g_{00}))h_0, h_0 - h_1\rangle = \langle(\pi_1(g_{00}) \otimes \pi_2(g_{10}))h_0, h_0 - h_1\rangle = \sqrt{2}\,, \tag{5.2.6}$$

Set $\eta_0 = \frac{1}{\sqrt{2}}(h_0 + h_1)$ and $\eta_1 = \frac{1}{\sqrt{2}}(h_0 - h_1)$. Then η_0 and η_1 are orthonormal because h_0 and h_1 are orthonormal. Equations (5.2.5) and (5.2.6) imply that

$$(\pi_1(g_{00}) \otimes \pi_2(g_{00}))h_0 = (\pi_1(g_{10}) \otimes \pi_2(g_{10}))h_0 = \eta_0\,, \tag{5.2.7}$$

$$(\pi_1(g_{10}) \otimes \pi_2(g_{00}))h_0 = (\pi_1(g_{00}) \otimes \pi_2(g_{10}))h_0 = \eta_1\,. \tag{5.2.8}$$

Similarly, by (5.2.4), we obtain

$$(\pi_1(g_{01}) \otimes \pi_2(g_{01}))h_1 = (\pi_1(g_{11}) \otimes \pi_2(g_{11}))h_1 = \eta_0\,,$$

$$(\pi_1(g_{01}) \otimes \pi_2(g_{11}))h_1 = (\pi_1(g_{01}) \otimes \pi_2(g_{11}))h_1 = \eta_1\,.$$

Thus, the vectors h_0, h_1, η_0 and η_1 can be converted to each other by local operations (i.e., tensor products of unitaries), which implies that h_0, h_1 and η_0 have the same Schmidt coefficients. On the other hand, consider the unitaries $X_1 = \pi_1(g_{10}g_{00}^*)$ and $X_2 = \pi_2(g_{10}g_{00}^*)$. Note that $h_0 = \frac{1}{\sqrt{2}}(\eta_0 + \eta_1)$ and $h_1 = \frac{1}{\sqrt{2}}(\eta_0 - \eta_1)$. Equations (5.2.7) and (5.2.8) show that

$$(X_1 \otimes 1)\eta_0 = (\pi_1(g_{10}g_{00}^*) \otimes \pi_2(g_{00}g_{00}^*))\eta_0 = (\pi_1(g_{10}) \otimes \pi_2(g_{00}))h_0 = \eta_1 \tag{5.2.9}$$

$$(X_1 \otimes 1)\eta_1 = (\pi_1(g_{10}g_{00}^*) \otimes \pi_2(g_{10}g_{10}^*))\eta_1 = (\pi_1(g_{10}) \otimes \pi_2(g_{10}))h_0 = \eta_0. \tag{5.2.10}$$

Similarly, $(1 \otimes X_2)\eta_0 = \eta_1$ and $(1 \otimes X_2)\eta_1 = \eta_0$. Combining linearity with equations (5.2.9) and (5.2.10), we obtain

$$(X_1 \otimes 1)h_0 = (1 \otimes X_2)h_0 = h_0 \;\text{ and }\; (X_1 \otimes 1)h_1 = (1 \otimes X_2)h_1 = -h_1\,.$$

Then

$$h_0 \in P_1\mathcal{H} \otimes P_2\mathcal{H}\,, h_1 \in (P_1\mathcal{H})^\perp \otimes (P_2\mathcal{H})^\perp\,,$$

where P_1 (resp. P_2) is the spectral projection of X_1 (resp. X_2) corresponding to the eigenvalue 1. In this situation, if the largest Schmidt coefficient of h_0 and h_1 is $\lambda_0 > 0$, then the largest Schmidt coefficient of η_0 is at most $\frac{1}{\sqrt{2}}\lambda_0$, which leads to a contradiction since this coefficient must be λ_0. □

Remark 5.2.2. It is clear from (5.2.2) that the relations (5.2.3)–(5.2.4) described in Theorem 5.2.1 can be represented by a non-spatial ucp map $\Omega : C^*(\mathbb{F}_4) \otimes_{\min} C^*(\mathbb{F}_4) \to M_2$. This fact can also be observed directly by using approximate embezzlement of entangled states from Theorem 3.3.3. Indeed, Theorem 3.3.3 shows that, for each $n \geq 1$, there exists a finite dimensional Hilbert space $\mathcal{H}_n$, a unit vector $\xi_n \in \mathcal{H}_n \otimes \mathcal{H}_n$, and unitary operators $U_n = (u_{jk}^n)_{j,k=0}^1$ on $\ell_2^2 \otimes \mathcal{H}_n$ and $V_n = (v_{jk}^n)_{j,k=0}^1$ on $\mathcal{H}_n \otimes \ell_2^2$ such that

$$\left\| (U_n \otimes V_n)(e_0 \otimes \xi_n \otimes e_0) - \frac{1}{\sqrt{2}}(e_0 \otimes \xi_n \otimes e_0 + e_1 \otimes \xi_n \otimes e_1) \right\|_{\ell_2^2 \otimes \mathcal{H}_n \otimes \mathcal{H}_n \otimes \ell_2^2} \leq \frac{1}{n}\,. \quad (5.2.11)$$

By the universal property of $C^*(\mathbb{F}_4)$, we may choose a sequence of unital $*$-homomorphisms $\pi_1^n : C^*(\mathbb{F}_4) \to M_2(\mathcal{B}(\mathcal{H}_n))$ such that

$$\begin{aligned}
\pi_1^n(g_{00}) &= U_n = \begin{pmatrix} u_{00}^n & u_{01}^n \\ u_{10}^n & u_{11}^n \end{pmatrix}, \\
\pi_1^n(g_{10}) &= (X \otimes 1)U_n = \begin{pmatrix} u_{00}^n & u_{01}^n \\ -u_{10}^n & -u_{11}^n \end{pmatrix}, \\
\pi_1^n(g_{01}) &= (X \otimes 1)U_n(Z \otimes 1) = \begin{pmatrix} u_{01}^n & u_{00}^n \\ -u_{11}^n & -u_{10}^n \end{pmatrix}, \\
\pi_1^n(g_{11}) &= U_n(Z \otimes 1) = \begin{pmatrix} u_{01}^n & u_{00}^n \\ u_{11}^n & u_{10}^n \end{pmatrix},
\end{aligned}$$

where X and Z are the Pauli matrices in M_2. Similarly, there is a sequence of unital $*$-homomorphisms $\pi_2^n : C^*(\mathbb{F}_4) \to M_2(\mathcal{B}(\mathcal{H}_n))$ such that

$$\begin{aligned}
\pi_2^n(g_{00}) &= V_n, \\
\pi_2^n(g_{10}) &= (X \otimes 1)V_n, \\
\pi_2^n(g_{01}) &= (X \otimes 1)V_n(Z \otimes 1), \\
\pi_2^n(g_{11}) &= V_n(Z \otimes 1).
\end{aligned}$$

Set $h_0^n = e_0 \otimes \xi_n \otimes e_0$ and $h_1^n = e_1 \otimes \xi_n e_1$. Define $W_n : \ell_2^2 \to \ell_2^2 \otimes \mathcal{H}_n \otimes \mathcal{H}_n \otimes \ell_2^2$ to be the isometry given by $W_n e_j = h_j^n$ for $j = 0, 1$. By (5.2.11), we can choose Ω as a weak*-limit point of the ucp maps

$$\Omega_n(\cdot) = W_n^*(\pi_1^n \otimes \pi_2^n)(\cdot)W_n\,.$$

Then it is readily checked that Ω satisfies the constraints of equation (5.2.2).

A modification of Theorem 5.2.1 allows us to find a spatial M_2-valued ucp map on $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2)$ that cannot be witnessed on a tensor product of finite-dimensional spaces.

Lemma 5.2.3. *Let $\mathcal{H}$ be a finite-dimensional Hilbert space. Then there do not exist unitaries u_0, u_1, v_0, v_1 on $\mathcal{H}$ and orthonormal vectors $h_0, h_1 \in \mathcal{H} \otimes \mathcal{H}$ satisfying*

$$\langle (u_j \otimes v_k)h_0, h_0\rangle = \frac{1}{\sqrt{2}} \ , \ \langle (u_j \otimes v_k)h_0, h_1\rangle = \frac{(-1)^{j-k}}{\sqrt{2}} \ , \ \ 0 \le j,k \le 1 \ . \tag{5.2.12}$$

Proof. Suppose that $\mathcal{H}$ is a finite-dimensional Hilbert space; u_0, u_1, v_0, v_1 are unitaries in $\mathcal{H}$; and $h_0, h_1 \in \mathcal{H}$ are unit vectors satisfying equation (5.2.12). As in the proof of Theorem 5.2.1, equation (5.2.12) implies that

$$\begin{aligned}(u_0 \otimes v_0)h_0 = (u_1 \otimes v_1)h_0 &= \frac{1}{\sqrt{2}}(h_0 + h_1) \ , \\ (u_0 \otimes v_1)h_0 = (u_1 \otimes v_0)h_0 &= \frac{1}{\sqrt{2}}(h_0 - h_1),\end{aligned} \tag{5.2.13}$$

and

$$h_0 \in P_1\mathcal{H} \otimes P_2\mathcal{H} \ , h_1 \in (P_1\mathcal{H})^\perp \otimes (P_2\mathcal{H})^\perp \ . \tag{5.2.14}$$

where P_1 (respectively, P_2) is the projection of $\mathcal{H}$ onto the eigenspace of $u_1u_0^*$ (respectively, $v_1v_0^*$) corresponding to the eigenvalue 1. Since $\mathcal{H}$ is finite dimensional, the Schmidt ranks of h_0, h_1, and $\frac{1}{\sqrt{2}}(h_0 + h_1)$ are all finite and nonzero. Equation (5.2.14) implies that the Schmidt rank of $\frac{1}{\sqrt{2}}(h_0 + h_1)$ is sum of the Schmidt ranks of h_0 and h_1. On the other hand, it follows from (5.2.13) that h_0 and $\frac{1}{\sqrt{2}}(h_0 + h_1)$ have the same Schmidt rank, which is a contradiction. $\square$

It is not hard to see that Lemma 5.2.3 still holds when we replace $\mathcal{H} \otimes \mathcal{H}$ with $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional. Indeed, if $\dim(\mathcal{H}_A) > \dim(\mathcal{H}_B)$, then we enlarge $\mathcal{H}_B$ to have the same dimension as $\mathcal{H}_A$, and we extend the unitaries to $\mathcal{H}_B^\perp$ by defining them to be the identity on $\mathcal{H}_B^\perp$. Then the claim of Lemma 5.2.3 still holds.

Conversely, if $\mathcal{H}$ is infinite-dimensional, then there always exist unitaries and unit vectors satisfying equation (5.2.12). For our purposes, we only need the following explicit example.

Lemma 5.2.4. *Let $\mathcal{H} = \ell^2(\mathbb{Z})$, and let T, U be unitaries on $\mathcal{H}$ given by*

$$Te_j = \begin{cases} e_j & j < 0 \\ -e_j & j \ge 0 \end{cases} \quad \textit{and} \quad Ue_j = e_{j+1}. \tag{5.2.15}$$

Define unit vectors ζ_1, ζ_2 in $\mathcal{H} \otimes \mathcal{H}$ via

$$\zeta_1 = \sum_{j<0} (\sqrt{2})^j e_j \otimes e_j \quad \text{and} \quad \zeta_2 = e_0 \otimes e_0. \tag{5.2.16}$$

Then setting $u_0 = v_0 = U$ and $u_1 = v_1 = TU$ yields equation (5.2.12).

Proof. It is not hard to see that

$$(U \otimes U)\zeta_1 = \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2),$$

while

$$(T \otimes 1)\zeta_1 = (1 \otimes T)\zeta_1 = \zeta_1$$

and

$$(T \otimes 1)\zeta_2 = (1 \otimes T)\zeta_2 = -\zeta_2.$$

Then setting $u_0 = v_0 = U$ and $u_1 = v_1 = TU$, we obtain the desired result. □

Lemma 5.2.5. *Let $\mathcal{H}$ be a Hilbert space.*

(i) There are no unitaries $u_0, u_1, u_2, v_0, v_1, v_2 \in \mathcal{B}(\mathcal{H})$ along with orthonormal vectors $h_0, h_1 \in \mathcal{H} \otimes \mathcal{H}$ satisfying equation (5.2.12) *and the additional condition that*

$$(u_2 \otimes v_2)h_0 = h_1 \; . \tag{5.2.17}$$

(ii) There exists a ucp map $\Psi : C^*(\mathbb{F}_3) \otimes_{min} C^*(\mathbb{F}_3) \to M_2$ such that

$$\Psi(g_0 \otimes g_0) = \begin{bmatrix} * & * \\ 1 & * \end{bmatrix} \; , \; \Psi(g_j \otimes g_k) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{(-1)^{j-k}}{\sqrt{2}} & * \end{bmatrix} \; , \; 1 \leq j, k \leq 2 \; . \tag{5.2.18}$$

Moreover, any ucp map that satisfies (5.2.18) *is not spatial.*

Proof. To see that (i) holds, we observe that conditions (5.2.12) and (5.2.17) imply that h_0, h_1 and $\frac{1}{\sqrt{2}}(h_0 + h_1)$ have the same Schmidt coefficients. Combined with (5.2.13) and (5.2.14), this leads to the same contradiction as in Theorem 5.2.1.

To show that (ii) is true, we use approximate embezzlement. With the same notation as in Remark 5.2.2, we define $\pi_1^n, \pi_2^n : C^*(\mathbb{F}_3) \to M_2(\mathcal{B}(\mathcal{H}_n))$ by

$$\begin{aligned} &\pi_1^n(g_0) = Z \otimes 1 \; , \; \pi_1^n(g_1) = U_n \; , \; \pi_1^n(g_2) = (X \otimes 1)U_n \; , \\ &\pi_2^n(g_0) = Z \otimes 1 \; , \; \pi_2^n(g_1) = V_n \; , \; \pi_2^n(g_2) = (X \otimes 1)V_n \; . \end{aligned} \tag{5.2.19}$$

Define $h_0^n = e_0 \otimes \xi_n \otimes e_0$ and $h_1^n = e_1 \otimes \xi_n \otimes e_1$. Note that $(\pi_1^n(g_0) \otimes \pi_2^n(g_0))h_0^n = h_1^n$ for all n. We let W_n be the isometry from Remark 5.2.2. Then let Ψ be a point-norm cluster point of the sequence of ucp maps $\Psi_n(\cdot) = W_n^*(\pi_1^n \otimes \pi_2^n(\cdot))W_n$. Then Ψ is ucp and satisfies the constraints of equation 5.2.18. Moreover, any such Ψ is not spatial, otherwise the map in (i) would be spatial. This completes the proof. $\square$

Remark 5.2.6. In Lemma 5.2.5(ii), the non-spatial ucp map $\Psi : C^*(\mathbb{F}_3)\otimes_{min}C^*(\mathbb{F}_3) \to M_2$ induces a non-spatial ucp map $\widetilde{\Psi} : C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}) \to M_2$. This is because, in the approximation in equation (5.2.19), the operators $\pi_1^n(g_0), \pi_2^n(g_0), \pi_1^n(g_1g_2^{-1})$ and $\pi_2^n(g_1g_2^{-1})$ are self-adjoint unitaries for every n. Then the representation $\pi_n = \pi_1^n \otimes \pi_2^n$ can also be defined on $C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z})$, since $C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z})$ is the universal C^*-algebra generated by two self-adjoint unitaries σ_0, σ_1 and one free unitary g. In this case, the corresponding representations $\rho_1^n, \rho_2^n : C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}) \to M_2(\mathcal{B}(\mathcal{H}_n))$ are given by

$$\rho_i^n(\sigma_0) = \pi_i^n(g_0),\ \rho_i^n(\sigma_1) = \pi_i^n(g_1g_2^{-1}),\ \rho_i^n(g) = \pi_i^n(g_2).$$

In the limit as $n \to \infty$, the relations in (5.2.18) translate to

$$\begin{aligned} &\widetilde{\Psi}(\sigma_0 \otimes \sigma_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \widetilde{\Psi}(g \otimes g) = \widetilde{\Psi}(\sigma_1 g \otimes \sigma_1 g) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{1}{\sqrt{2}} & * \end{bmatrix}, \\ &\widetilde{\Psi}(\sigma_1 g \otimes g) = \widetilde{\Psi}(g \otimes \sigma_1 g) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{-1}{\sqrt{2}} & * \end{bmatrix}. \end{aligned} \tag{5.2.20}$$

This ucp map leads to the following theorem, from which Theorem 5.0.5 follows. Using our methods, the smallest matrix size n for which we obtain a separation $C_{qs}^{(n)}(m,k) \neq C_{qa}^{(n)}(m,k)$ for some m, k is $n = 3$ (see part (iii) of the following theorem).

Theorem 5.2.7. *We have the following separations.*

(i) $F_q^{(2)}(2) \neq F_{qs}^{(2)}(2)$;

(ii) $F_{qs}^{(2)}(3) \neq F_{qa}^{(2)}(3)$;

(iii) $C_{qs}^{(3)}(4,2) \neq C_{qa}^{(3)}(4,2)$;

(iv) $C_{qs}^{(5)}(3,2) \neq C_{qa}^{(5)}(3,2)$;

(v) $C_{qs}^{(13)}(2,3) \neq C_{qa}^{(13)}(2,3)$.

Proof. Let g_1, g_2 be a set of universal generators of $C^*(\mathbb{F}_2)$ and let g_0, g_1, g_2 be a set of universal generators of $C^*(\mathbb{F}_3)$. It is a direct consequence of Lemmas 5.2.3 and 5.2.4 that the assignment

$$\Psi(g_j \otimes g_k) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{(-1)^{j-k}}{\sqrt{2}} & * \end{bmatrix}, \ 1 \leq j, k \leq 2$$

represents a spatial M_2-valued correlation in $F_{qs}^{(2)}(2)$ which does not belong to $F_q^{(2)}(2)$. Similarly, the separation $F_{qs}^{(2)}(3) \neq F_{qa}^{(2)}(3)$ follows from Lemma 5.2.5 (ii).

To show that $C_{qs}^{(3)}(4,2) \neq C_{qa}^{(3)}(4,2)$, we consider the embedding of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ into $*_4\mathbb{Z}_2$ from Proposition 1.4.3. This embedding is given by

$$\sigma_0 \mapsto \sigma_0 \ , \ \sigma_1 \mapsto \sigma_1 \ , \ g \mapsto \sigma_2\sigma_3 \ ,$$

where $\sigma_0, ..., \sigma_3$ are generators of $\mathbb{Z}_2$ and g is the generator of $\mathbb{Z}$. By Proposition 1.4.1, this embedding induces a C^*-algebraic embedding $C^*(\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}) \hookrightarrow C^*(*_4\mathbb{Z}_2)$, and the non-spatial correlation in Remark 5.2.6 extends to an M_2-valued ucp map on $C^*(*_4\mathbb{Z}_2) \otimes_{\min} C^*(*_4\mathbb{Z}_2)$. This ucp map satisfies the following:

$$\widetilde{\Psi}(\sigma_2\sigma_3 \otimes \sigma_2\sigma_3) = \widetilde{\Psi}(\sigma_1\sigma_2\sigma_3 \otimes \sigma_1\sigma_2\sigma_3) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.21}$$

$$\widetilde{\Psi}(\sigma_1\sigma_2\sigma_3 \otimes \sigma_2\sigma_3) = \widetilde{\Psi}(\sigma_2\sigma_3 \otimes \sigma_1\sigma_2\sigma_3) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{-1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.22}$$

$$\widetilde{\Psi}(\sigma_0 \otimes \sigma_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.2.23}$$

Let $\mathcal{P} = \text{span}\ \{1\} \cup \{\sigma_i\}_{i=0}^3$; we want to obtain a matrix-valued non-spatial correlation defined only on the operator system $\mathcal{P} \otimes \mathcal{P}$ in $C^*(*_4\mathbb{Z}_2) \otimes_{\min} C^*(*_4\mathbb{Z}_2)$, and not on products of elements of $\mathcal{P} \otimes \mathcal{P}$. To accomplish this, we add extra dimensions to the output space. Let $\widetilde{\Phi}(\cdot) = V^*\pi(\cdot)V$ be a Stinespring dilation of $\widetilde{\Phi}$, where π is a unital representation of $C^*(*_4\mathbb{Z}_2) \otimes_{\min} C^*(*_4\mathbb{Z}_2)$ on some Hilbert space $\mathcal{H}$ and $V : \ell_2^2 \to \mathcal{H}$ is an isometry. Define unit vectors

$$h_0 := Ve_0 \ , \ h_1 := Ve_1, \ h_2 := \pi(\sigma_3 \otimes \sigma_3)h_0,$$

and the operator

$$W : \ell_2^3 \to \mathcal{H} \ , \ We_j = h_j \ , \ 0 \leq j \leq 2 \ .$$

Then $\Psi(\cdot) = W^*\pi(\cdot)W$ gives an M_3-valued, possibly non-unital completely positive map. Let $\eta_0 = \frac{1}{\sqrt{2}}(h_0 + h_1)$ and $\eta_1 = \frac{1}{\sqrt{2}}(h_0 - h_1)$. We note that

$$\pi(\sigma_2\sigma_3 \otimes \sigma_2\sigma_3)h_0 = \pi(\sigma_1\sigma_2\sigma_3 \otimes \sigma_1\sigma_2\sigma_3)h_0 = \eta_0,$$

while

$$\pi(\sigma_1 \otimes 1)\eta_0 = \pi(1 \otimes \sigma_1)\eta_0 = \eta_1 \text{ and } \pi(\sigma_1 \otimes \sigma_1)\eta_0 = \eta_0. \tag{5.2.24}$$

Similarly, since $\pi(\sigma_1 \otimes 1)$, $\pi(1 \otimes \sigma)$ and $\pi(\sigma_1 \otimes \sigma_1)$ are all self-adjoint, we have

$$\pi(\sigma_1 \otimes 1)\eta_1 = \pi(1 \otimes \sigma_1)\eta_1 = \eta_0 \text{ and } \pi(\sigma_1 \otimes \sigma_1)\eta_1 = \eta_1. \tag{5.2.25}$$

Since $h_0 = \frac{1}{\sqrt{2}}(\eta_0 + \eta_1)$ and $h_1 = \frac{1}{\sqrt{2}}(\eta_0 - \eta_1)$, by linearity we obtain $\pi(\sigma_1 \otimes 1)h_0 = \pi(1 \otimes \sigma_1)h_0 = h_1$. Thus, $\pi(\sigma_1 \otimes \sigma_1)h_j = h_j$ for $j = 0, 1$. Since $\langle h_0, h_1\rangle = 0$, the upper-left 2×2 block of Ψ on each generator must be a contraction. Using the fact that $\Psi(\sigma_i \otimes \sigma_j)$, $\Psi(1 \otimes \sigma_i)$ and $\Psi(\sigma_i \otimes 1)$ are self-adjoint for all $0 \leq i \leq 3$, equations (5.2.21)–(5.2.25) show that

$$\Psi(\sigma_0 \otimes \sigma_0) = \begin{bmatrix} 0 & 1 & * \\ 1 & 0 & * \\ * & * & * \end{bmatrix}, \ \Psi(\sigma_2 \otimes \sigma_2) = \frac{1}{\sqrt{2}} \begin{bmatrix} * & * & 1 \\ * & * & 1 \\ 1 & 1 & * \end{bmatrix}, \ \Psi(\sigma_3 \otimes \sigma_3) = \begin{bmatrix} * & * & 1 \\ * & * & * \\ 1 & * & * \end{bmatrix} \tag{5.2.26}$$

$$\Psi(\sigma_1 \otimes \sigma_1) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ * & * & * \end{bmatrix}, \ \Psi(\sigma_1 \otimes 1) = \begin{bmatrix} 1 & 0 & * \\ 0 & -1 & * \\ * & * & * \end{bmatrix}, \ \Psi(1 \otimes \sigma_1) = \begin{bmatrix} 1 & 0 & * \\ 0 & -1 & * \\ * & * & * \end{bmatrix}, \tag{5.2.27}$$

Then any cp map $\Theta : C^*(*_4\mathbb{Z}_2) \otimes_{min} C^*(*_4\mathbb{Z}_2) \to M_3$ that coincides with Ψ on $\mathcal{P} \otimes \mathcal{P}$ will satisfy equations (5.2.26) and (5.2.27), and hence cannot be spatial. Thus, we have shown that $\Psi|_{\mathcal{P}\otimes\mathcal{P}}$ is a non-spatial correlation in the non-unital context. To obtain a non-spatial ucp map, we let Ψ' be a point-norm cluster point of the net of ucp maps given by $\Psi_\varepsilon(\cdot) = (\Psi(1) + \varepsilon I_3)^{-\frac{1}{2}}\Psi(\cdot)(\Psi(1) + \varepsilon I_3)^{-\frac{1}{2}}$ for $\varepsilon > 0$. Then the restriction $\Psi'|_{\mathcal{P}\otimes\mathcal{P}}$ gives an M_3-valued ucp map which is not spatial. This is because if Θ' is a spatial ucp map and $\Theta'|_{\mathcal{P}\otimes\mathcal{P}} = \Psi'|_{\mathcal{P}\otimes\mathcal{P}}$, then the map $\widetilde{\Theta}(\cdot) = \Psi(1)^{1/2}\Theta'(\cdot)\Psi(1)^{1/2}$ will be a spatial cp map that agrees with Ψ on $\mathcal{P} \otimes \mathcal{P}$, which leads to a contradiction. Hence, we obtain a correlation in $C_{qa}^{(3)}(4, 2)$ which is not in $C_{qs}^{(3)}(4, 2)$.

For (iv), we consider the following group embedding of $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}$ into $*_3\mathbb{Z}_2$ from Proposition 1.4.4:

$$\sigma_0 \mapsto \sigma_0 \ , \ \sigma_1 \mapsto \sigma_1 \ , \ g \mapsto \sigma_2\sigma_0\sigma_1\sigma_2 \ .$$

Let $\omega = \sigma_2\sigma_0\sigma_1\sigma_2$. Then the non-spatial correlation in Remark 5.2.6 extends to a ucp map on $C^*(*_3\mathbb{Z}_2) \otimes_{\min} C^*(*_3\mathbb{Z}_2)$ as follows:

$$\widetilde{\Phi}(\omega \otimes \omega) = \widetilde{\Phi}(\sigma_1\omega \otimes \sigma_1\omega) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.28}$$

$$\widetilde{\Phi}(\sigma_1\omega \otimes \omega) = \widetilde{\Phi}(\omega \otimes \sigma_1\omega) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{-1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.29}$$

$$\widetilde{\Phi}(\sigma_0 \otimes \sigma_0) = \begin{bmatrix} * & * \\ 1 & * \end{bmatrix}, \tag{5.2.30}$$

Let $\widetilde{\Phi}(\cdot) = V_0^*\pi_0(\cdot)V_0$ be a Stinespring dilation of $\widetilde{\Phi}$, and let $k_j = V_0e_j$ for $0 \le j \le 1$. In order to express the above non-spatial correlation on the generators, we define three intermediate vectors

$$\begin{aligned} &k_2 := \pi_0(\sigma_2 \otimes \sigma_2)k_0,\ k_3 := \pi_0(\sigma_1 \otimes \sigma_1)k_2\ , \\ &k_4 := \pi_0(\sigma_0 \otimes \sigma_0)k_3\ . \end{aligned}$$

Then note that $\pi_0(\sigma_2 \otimes \sigma_2)k_4 = \pi_0(\omega \otimes \omega)k_0 = \frac{1}{\sqrt{2}}(k_0 + k_1)$. Thus, to describe $\widetilde{\Phi}$ in terms of a matrix-valued correlation defined only on the generators, it suffices to use the vectors $\{k_j\}_{j=0}^4$. In a similar manner to the argument for (iii), we obtain an M_5-valued non-spatial correlation in $C_{qs}^{(5)}(3,2)$.

The same argument leads to a matrix-valued non-spatial correlation for $\mathbb{Z}_3 * \mathbb{Z}_3$. Indeed, let a, b be the generators of the two copies of $\mathbb{Z}_3$. We use the embedding $\mathbb{F}_3 \to \mathbb{Z}_3 * \mathbb{Z}_3$ from Proposition 1.4.6:

$$\sigma_0 \mapsto aba,\ \sigma_1 \mapsto bab,\ g \mapsto ab^2a^2b\ .$$

Then the non-spatial correlation from Remark 5.2.6 extends to a ucp map Γ on $C^*(\mathbb{Z}_3 * \mathbb{Z}_3) \otimes_{\min} C^*(\mathbb{Z}_3 * \mathbb{Z}_3)$ via

$$\Gamma(ab^2a^2b \otimes ab^2a^2b) = \Gamma(bab \cdot ab^2a^2b \otimes bab \cdot ab^2a^2b) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.31}$$

$$\Gamma(bab \cdot ab^2a^2b \otimes ab^2a^2b) = \Gamma(ab^2a^2b \otimes bab \cdot ab^2a^2b) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ -\frac{1}{\sqrt{2}} & * \end{bmatrix}, \tag{5.2.32}$$

$$\Gamma(aba \otimes aba) = \begin{bmatrix} * & * \\ 1 & * \end{bmatrix}. \tag{5.2.33}$$

We want a matrix-valued, non-spatial ucp map specified on $\mathcal{R}\otimes\mathcal{R}$, where

$$\mathcal{R} = \text{span }\{1,a,b,a^*,b^*\} = \text{span }\{1,a,b,a^2,b^2\} \subseteq C^*(\mathbb{Z}_3 * \mathbb{Z}_3).$$

Let $\Gamma(\cdot) = V_1^*\pi_1 V_1$ be a Stinespring dilation for Γ and $\zeta_j = V_1 e_j$ for $j = 0,1$. We define eleven intermediate vectors as follows:

$$\zeta_2 = \pi_1(a\otimes a)\zeta_0,\ \zeta_3 = \pi_1(b\otimes b)\zeta_2, \tag{5.2.34}$$

$$\zeta_4 = \pi_1(b\otimes b)\zeta_0,\ \zeta_5 = \pi_1(a^2\otimes a^2)\zeta_4,\ \zeta_6 = \pi_1(b^2\otimes b^2)\zeta_5. \tag{5.2.35}$$

Setting $\theta_0 = \frac{1}{\sqrt{2}}(\zeta_0+\zeta_1)$ and $\theta_1 = \frac{1}{\sqrt{2}}(\zeta_0-\zeta_1)$. It follows that

$$\pi_1(a\otimes a)\zeta_6 = \pi_1(ab^2a^2b\otimes ab^2a^2b)\zeta_0 = \theta_0 \tag{5.2.36}$$

Define

$$\zeta_7 = \pi_1(1\otimes b)\theta_0,\ \zeta_8 = \pi_1(1\otimes a)\zeta_7, \tag{5.2.37}$$

$$\zeta_9 = \pi_1(b\otimes 1)\theta_0,\ \zeta_{10} = \pi_1(a\otimes 1)\zeta_9, \tag{5.2.38}$$

$$\zeta_{11} = \pi_1(b\otimes b)\theta_0,\ \zeta_{12} = \pi_1(a\otimes a)\zeta_{11}. \tag{5.2.39}$$

The definition of Γ can be summarized in terms of the vectors via

$$\pi_1(a\otimes a)\zeta_6 = \pi_1(b\otimes b)\zeta_{12} = \theta_0, \tag{5.2.40}$$

$$\pi_1(b\otimes 1)\theta_0 = \pi_1(1\otimes b)\theta_0 = \theta_1, \tag{5.2.41}$$

$$\pi_1(a\otimes a)\zeta_3 = \zeta_1. \tag{5.2.42}$$

By the same argument as for (iii) and (iv), equations (5.2.34) through (5.2.42) show that $C_{qs}^{(13)}(2,3)$ is not closed, which completes the proof. □

Note that for $(d,m) = (2,2)$, $\mathbb{Z}_2 * \mathbb{Z}_2$ is an amenable group isomorphic to the semi-direct product $\mathbb{Z}\rtimes\mathbb{Z}_2$, and its irreducible representations are at most 2-dimensional. Thus, our result is optimal in the sense that for $(d,m) = (2,2)$, $C_q^{(n)}(2,2) = C_{qs}^{(n)}(2,2) = C_{qa}^{(n)}(2,2) = C_{qc}^{(n)}(2,2)$ for all n. We have shown that for all non-trivial sizes $(d,m) \neq (2,2)$, the spatial quantum correlation set in d inputs and m outputs is not closed at some matrix level. The question of whether scalar-valued spatial quantum correlation sets are closed for sizes smaller than $(5,2)$ remains open.

For the separation between C_q and C_{qs}, Coladangelo and Stark recently proved in [12] that $C_q(5,3) \neq C_{qs}(5,3)$, which separates the set of scalar-valued correlations on finite dimensional Hilbert spaces and the set of scalar-valued correlations on tensor products of (possibly infinite dimensional) Hilbert spaces. In Chapter 6, we will apply Lemma 5.2.4 to obtain the separations $C_q^{(4)}(2,3) \neq C_{qs}^{(4)}(2,3)$ and $C_q^{(3)}(3,2) \neq C_{qs}^{(3)}(3,2)$.

Remark 5.2.8. Let G_1 and G_2 be two finite groups. It is known that the product $G_1 * G_2$ contains a copy of $\mathbb{F}_2$ if and only if $|G_1| + |G_2| \geq 5$ and $|G_1|, |G_2| > 1$ (see, for example, [58, p. 8]). Using this kind of group embedding, we know that there exists an n such that the M_n-valued spatial correlation set for $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ is not closed (or for any $C^*(G_1 * G_2)$).

The main idea we used in the proof of Theorem 5.2.7 is to reduce the length of the words in the definition of the ucp map by adding intermediate vectors. Conversely, we can reduce the matrix size for the ucp map by allowing correlations that use words with length greater than 1. Such quantum correlations on words are called *spatiotemporal correlations* in [25]. In the context of spatiotemporal correlations, we have the following observation.

Corollary 5.2.9. *Let g_0, g_1, g_2 be a set of universal generators of $C^*(\mathbb{F}_3)$. There exists a state ψ on $C^*(\mathbb{F}_3) \otimes_{min} C^*(\mathbb{F}_3)$ satisfying*

$$\psi(g_0 \otimes g_0) = 0 \ , \ \psi(g_j \otimes g_k) = \frac{1}{\sqrt{2}} \ , \ \psi(g_0 g_j \otimes g_0 g_k) = \frac{(-1)^{j-k}}{\sqrt{2}} \ , \ 1 \leq j, k \leq 2 \ . \tag{5.2.43}$$

Moreover, any state satisfying (5.2.43) *is not spatial.*

Proof. Let $\Psi : C^*(\mathbb{F}_3) \otimes_{min} C^*(\mathbb{F}_3) \to M_2$ be the UCP map from Lemma (5.2.5) satisfying

$$\Psi(g_0 \otimes g_0) = \begin{bmatrix} * & * \\ 1 & * \end{bmatrix} \ , \ \Psi(g_j \otimes g_k) = \begin{bmatrix} \frac{1}{\sqrt{2}} & * \\ \frac{(-1)^{j-k}}{\sqrt{2}} & * \end{bmatrix} \ , \ 1 \leq j, k \leq 2 \ . \tag{5.2.44}$$

Because Ψ is ucp and $g_0 \otimes g_0$ is a contraction, we know that $\Psi(g_0 \otimes g_0) = \begin{bmatrix} 0 & * \\ 1 & 0 \end{bmatrix}$. Then it is easy to see that ψ is given by the first diagonal entry of Ψ. Conversely, suppose such a state ψ is spatially implemented by

$$\psi(\cdot) = \langle ((\pi_1 \otimes \pi_2)(\cdot)) h_0, h_0 \rangle \ .$$

Let $h_1 := (\pi_1(g_0) \otimes \pi_2(g_0)) h_0$. Then h_1 is orthogonal to h_0 because

$$\langle h_1, h_0 \rangle = \langle (\pi_1(g_0) \otimes \pi_2(g_0)) h_0, h_0 \rangle = \psi(g_0 \otimes g_0) = 0 \ .$$

Then the vectors h_0 and h_1, together with the unitaries $\pi_1(g_j), \pi_2(g_j), 0 \leq j \leq 2$ give exactly the scenario as in Lemma 5.2.5 (i), which is a contradiction. $\square$

The point of the above corollary is that the non-spatial nature of the ucp map can be witnessed on words with length at most 2, which span a finite dimensional subspace. Although the set of spatial states always forms a weak*-dense subset of the state space of the minimal tensor product of two C^*-algebras, in general there are a plethora of non-spatial states for the minimal tensor product. For example, let $C[0,1]$ be the C^*-algebra of continuous functions on the unit interval. Then the minimal tensor $C[0,1] \otimes_{min} C[0,1] \cong C([0,1]^2)$ is the C^*-algebra of continuous functions on the unit square. A spatial state corresponds to a probability measure on $[0,1]^2$ that can be written as a (possibly infinite) convex combination of product probability measures. This is obviously not the case for the bilinear form

$$(f,g) \longrightarrow \int_{[0,1]} f(x)g(x)dx,$$

where dx is the Lebesgue measure. In general, spatial states on $\mathcal{A} \otimes_{\min} \mathcal{B}$ correspond to nuclear operators from $\mathcal{A}$ to $\mathcal{B}^*$. From the above example, non-spatial states exist for $\mathcal{A} \otimes_{\min} \mathcal{B}$ whenever both $\mathcal{A}$ and $\mathcal{B}$ contain operators with continuous spectra, because non-spatial states are extendable to larger algebras. Nevertheless, non-spatial states witnessed on finite dimensional subsystems can only exist when $\mathcal{A}$ and $\mathcal{B}$ are non-commutative.

Proposition 5.2.10. *Let $\mathcal{A}$ be a unital, commutative C^*-algebra, and let $\mathcal{B}$ be a unital C^*-algebra. Let $E \subset \mathcal{A}$ and $F \subset \mathcal{B}$ be finite dimensional operator systems. If φ is a state on $\mathcal{A} \otimes_{\min} \mathcal{B}$, then there exists a spatial state ψ on $\mathcal{A} \otimes_{\min} \mathcal{B}$ such that $\varphi|_{E\otimes F} = \psi|_{E\otimes F}$.*

Proof. The restriction $\varphi|_{E\otimes F}$ corresponds to a linear map $T : E \to F^*$ given by

$$T(x)(y) = \varphi(x \otimes y)\ , \forall\, x \in E, y \in F\ .$$

Since φ is a state, it is clear that $\|T(x)(y)\| \leq \|x\|\|y\|$, so that $\|T(x)\| \leq \|x\|$. Hence, $\|T\| = 1$. Because $\mathcal{A}$ is commutative, $E \otimes_{\min} F$ is isometric to the Banach space injective tensor product $E\widehat{\otimes}_{\epsilon}F$ by Theorem 1.2.12. Thus, φ has norm 1 when considered as a functional on $E\widehat{\otimes}_{\varepsilon}F$. This fact implies that the integral norm of T is 1. Moreover, by Theorem 1.2.8, the nuclear norm of T is also 1, since E and F are finite-dimensional. By definition of the nuclear norm, there exist functionals $\alpha_j \in E^*, \beta_j \in F^*$ and scalars $\lambda_j > 0$ for $1 \leq j \leq n$ with $\sum_{j=1}^{n} \lambda_j = 1$ such that

$$\varphi|_{E\otimes F} = \sum_{j=1}^{n} \lambda_j \alpha_j \otimes \beta_j$$

and $\| \alpha_j \|_{E^*} = \| \beta_j \|_{F^*} = 1$ for all j. Since $1_{\mathcal{A}} \otimes 1_{\mathcal{B}} \in E \otimes F$, we see that

$$\sum_{j=1}^{n} \lambda_j \alpha_j(1_{\mathcal{A}})\beta_j(1_{\mathcal{B}}) = \varphi(1_{\mathcal{A}} \otimes 1_{\mathcal{B}}) = 1.$$

Since 1 is an extreme point in the closed unit disk in $\mathbb{C}$, each α_j and β_j must be unital. In particular, each α_j is a state on E, and each β_j is a state on F. Let $\widetilde{\alpha}_j$ (respectively, $\widetilde{\beta}_j$) be a state on $\mathcal{A}$ (respectively, $\mathcal{B}$) that extends α_j (respectively, β_j). Thus we have that

$$\psi = \sum_j \lambda_j \widetilde{\alpha}_j \otimes \widetilde{\beta}_j$$

is a spatial state on $\mathcal{A} \otimes_{min} \mathcal{B}$ satisfying $\psi_{|E\otimes F} = \varphi_{|E\otimes F}$. $\square$

Chapter 6

Bipartite matrix-valued tensor product correlations that are not finitely representable

In Chapter 5, it was shown that $C_{qs}^{(5)}(3,2) \neq C_{qa}^{(5)}(3,2)$ and $C_{qs}^{(13)}(2,3) \neq C_{qa}^{(13)}(2,3)$. In this chapter, we prove analogous separations between the q and qs models.

Theorem 6.0.11. *For any $m, k \in \mathbb{N}$ with $m, k \geq 2$, $(m,k) \neq (2,2)$, there is $n \leq 4$ such that $C_q^{(n)}(m,k) \neq C_{qs}^{(n)}(m,k)$.*

In particular, in Theorem 6.1.1 we prove that $C_q^{(3)}(3,2) \neq C_{qs}^{(3)}(3,2)$, and in Theorem 6.2.2 we prove that $C_q^{(4)}(2,3) \neq C_{qs}^{(4)}(2,3)$. Our methods draw on using an explicit tensor product of unitary representations of the group $\mathbb{Z}_2 * \mathbb{Z}$ and associated behaviour that cannot be witnessed on finite-dimensional Hilbert spaces. We use certain facts about group embeddings regarding free groups to translate these representations into the context of matrix-valued correlations. The interested reader can see [16] for more information on these group embeddings.

In the case when $n = 1$, the only known separation of $C_q(m,k)$ from $C_{qs}(m,k)$ is the recent result of A. Codalangelo and J. Stark, which says that $C_q(5,3) \neq C_{qs}(5,3)$ [12]. On the other hand, for smaller input and output sets, it is not known whether $C_q(m,k) = C_{qs}(m,k)$. It is widely thought that $C_q(3,2) \neq C_{qs}(3,2)$; indeed, it was conjectured by K.F. Pál and T. Vértesi [48] that the famous I_{3322} inequality should have a maximal violation in $C_{qs}(3,2)$ with no such violation in $C_q(3,2)$. If this conjecture were true, then it would imply that $C_q(3,2) \neq C_{qs}(3,2)$. Nonetheless, determining whether $C_q(m,k) \neq C_{qs}(m,k)$ for input and output sets smaller than the example of [12] remains open. However, if we allow for matrix-valued correlations, then by Theorem 6.0.11, an analogue of the separation $C_q(m,k) \neq C_{qs}(m,k)$ will hold for any pair (m,k) with $m,k \geq 2$ and $(m,k) \neq (2,2)$.

Recall that any matrix-valued tensor product correlation $(P(a,b|x,y))$ is of the form

$$P(a,b|x,y) = (\langle (E_{a,x} \otimes F_{b,y})\eta_j, \eta_i \rangle)_{i,j=1}^n,$$

where $\{E_{a,x}\}_{a=1}^k$ are PVMs on a Hilbert space $\mathcal{H}_A$, $\{F_{b,y}\}_{b=1}^k$ are PVMs on a Hilbert space $\mathcal{H}_B$, and $\eta_1, ..., \eta_n$ is a collection of orthonormal vectors in $\mathcal{H}_A \otimes \mathcal{H}_B$. Alternatively, we may associate to the vectors $\eta_1, ..., \eta_n$ the isometry $W : \mathbb{C}^n \to \mathcal{H}_A \otimes \mathcal{H}_B$ given by $We_i = \eta_i$ for all i. Then

$$P(a,b|x,y) = W^*(E_{a,x} \otimes F_{b,y})W.$$

Since $\sum_{b=1}^k F_{b,y} = I_{\mathcal{H}_B}$ for each $1 \leq y \leq m$, the matrix-valued marginal distribution for Alice is given by

$$P_A(a|x) = \sum_{b=1}^k P(a,b|x,y) = W^*(E_{a,x} \otimes I)W, \tag{6.0.1}$$

which is independent of y. Similarly, since $\sum_{a=1}^k E_{a,x} = I_{\mathcal{H}_A}$ for all $1 \leq x \leq m$, the matrix-valued marginal distribution for Bob is given by

$$P_B(b|y) = \sum_{a=1}^k P(a,b|x,y) = W^*(I \otimes F_{b,y})W, \tag{6.0.2}$$

which is independent of x. We will use these facts frequently in this chapter.

To work towards proving Theorems 6.1.1 and 6.2.2, we recall a result from Chapter 5. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ with canonical orthonormal basis $\{e_j\}_{j \in \mathbb{Z}}$, and let T, U be the unitaries given by

$$Te_j = \begin{cases} -e_j & j \geq 0 \\ e_j & j < 0 \end{cases} \quad \text{and} \quad Ue_j = e_{j+1}. \tag{6.0.3}$$

Then the unit vectors

$$\zeta_1 = \sum_{j<0} (\sqrt{2})^j e_j \otimes e_j \text{ and } \zeta_2 = e_0 \otimes e_0 \tag{6.0.4}$$

in $\mathcal{H} \otimes \mathcal{H}$, together with $T_1 = T_2 = T$ and $U_1 = U_2 = U$, yield the equations

$$\langle (U_1 \otimes U_2)\zeta_1, \zeta_2 \rangle = \langle (U_1 \otimes U_2)\zeta_1, \zeta_1 \rangle = \frac{1}{\sqrt{2}} \tag{6.0.5}$$

$$\langle (T_1 \otimes I)\zeta_1, \zeta_1 \rangle = \langle (I \otimes T_2)\zeta_1, \zeta_1 \rangle = 1 \tag{6.0.6}$$

$$\langle (T_1 \otimes I)\zeta_2, \zeta_2 \rangle = \langle (I \otimes T_2)\zeta_2, \zeta_2 \rangle = -1. \tag{6.0.7}$$

Moreover, by Lemma 5.2.3, equations (6.0.5) – (6.0.7) cannot be witnessed on a tensor product of finite-dimensional spaces. This example will be the basis for our spatial correlations that cannot be witnessed on finite-dimensional space.

Notice that, in the above example, T is a self-adjoint unitary, so that $T^2 = I$. Thus, the unitaries in equation (6.0.3) arise from a unital $*$-homomorphism $\pi : C^*(\mathbb{Z}_2 * \mathbb{Z}) \to \mathcal{B}(\ell^2(\mathbb{Z}))$ given by $\pi(\sigma) = T$ and $\pi(g) = U$, where $\mathbb{Z}_2 * \mathbb{Z}$ is the free product of the two-element group $\mathbb{Z}_2$ and the group of integers; σ is the generator of the copy of $\mathbb{Z}_2$ in $\mathbb{Z}_2 * \mathbb{Z}$, and g is a generator of $\mathbb{Z}$ in $\mathbb{Z}_2 * \mathbb{Z}$. Using group embeddings of $\mathbb{Z}_2 * \mathbb{Z}$ into $*_3\mathbb{Z}_2$ and $\mathbb{Z}_2 * \mathbb{Z}_3$ will allow us to translate Lemma 5.2.3 to separations of the form $C_q^{(n)}(m,k) \neq C_{qs}^{(n)}(m,k)$.

6.1 Three inputs and two outputs

In this section, we will exhibit a correlation in $C_{qs}^{(3)}(3,2)$ that is not in $C_q^{(3)}(3,2)$.

Theorem 6.1.1. *There exists an element $P = (P(a,b|x,y))_{a,b,x,y}$ in $C_{qs}^{(3)}(3,2)$ such that*

$$P(2,2|2,2) - P(1,2|2,2) - P(2,1|2,2) + P(1,1|2,2) = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{6.1.1}$$

$$P(2,2|3,3) - P(1,2|3,3) - P(2,1|3,3) + P(1,1|3,3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{6.1.2}$$

$$P_A(2|1) - P_A(1|1) = P_B(2|1) - P_B(1|1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{6.1.3}$$

Moreover, there is no element in $C_q^{(3)}(3,2)$ satisfying equations (6.1.1)–(6.1.3).

Proof. Let $\mathcal{H} = \ell^2(\mathbb{Z})$, and define unit vectors in $\mathcal{H} \otimes \mathcal{H}$ by

$$\zeta_1 = \sum_{j<0} (\sqrt{2})^j e_j \otimes e_j, \quad \zeta_2 = e_0 \otimes e_0, \quad \zeta_3 = \sum_{j>0} (\sqrt{2})^{-j} e_j \otimes e_j. \tag{6.1.4}$$

Let $\{f_1, f_2, f_3\}$ denote the canonical orthonormal basis for $\mathbb{C}^3$. It is easy to see that $\{\zeta_1, \zeta_2, \zeta_3\}$ is an orthonormal set, so that the map $W : \mathbb{C}^3 \to \mathcal{H} \otimes \mathcal{H}$ given by $W f_i = \zeta_i$ for $i = 1, 2, 3$ is an isometry. We define self-adjoint unitaries on the canonical basis vectors $\{e_j\}_{j \in \mathbb{Z}}$ of $\ell^2(\mathbb{Z})$ by

$$S_1 e_j = \begin{cases} -e_j & j \geq 0 \\ e_j & j < 0 \end{cases} \tag{6.1.5}$$

$$S_2 e_j = e_{-j+1} \tag{6.1.6}$$

$$S_3 e_j = e_{-j} \tag{6.1.7}$$

It is straightforward to check that each S_i is unitary and that $S_i^2 = I$, so that each S_i is a self-adjoint unitary. Thus, for each $x = 1, 2, 3$, there are PVM's $\{E_{1,x}, E_{2,x}\}$ on $\mathcal{H}$ such that $S_x = E_{2,x} - E_{1,x}$. We let $F_{b,y} = E_{b,y}$ for all b, y, and we define $P = (W^*(E_{a,x} \otimes F_{b,y})W)_{a,b,x,y}$, which is an element of $C_{qs}^{(3)}(3,2)$. We note that for each $x = 1, 2, 3$,

$$P(2,2|x,x) - P(2,1|x,x) - P(1,2|x,x) + P(1,1|x,x) = W^*(S_x \otimes S_x)W. \tag{6.1.8}$$

On the other hand,

$$P_A(2|1) - P_A(1|1) = W^*(S_1 \otimes I)W \text{ and } P_B(2|1) - P_B(1|1) = W^*(I \otimes S_1)W. \tag{6.1.9}$$

Using the unitaries defined above, it is routine to check that equations (6.1.1)–(6.1.3) are satisfied.

Now, suppose that $\widetilde{P} \in C_q^{(3)}(3,2)$ satisfies equations (6.1.1)–(6.1.3). Then there are finite-dimensional Hilbert spaces $\mathcal{K}_A$ and $\mathcal{K}_B$, PVM's $\{\widetilde{E}_{1,x}, \widetilde{E}_{2,x}\}$ on $\mathcal{K}_A$ for each $x = 1, 2, 3$, PVM's $\{\widetilde{F}_{1,y}, \widetilde{F}_{2,y}\}$ on $\mathcal{K}_B$ for each $y = 1, 2, 3$, and an isometry $V : \mathbb{C}^3 \to \mathcal{K}_A \otimes \mathcal{K}_B$ such that

$$\widetilde{P} = (V^*(\widetilde{E}_{a,x} \otimes \widetilde{F}_{b,y})V)_{a,b,x,y}.$$

Since $\widetilde{E}_{2,x} - \widetilde{E}_{1,x}$ is unitary, there is a unital $*$-homomorphism $\pi_A : C^*(*_3\mathbb{Z}_2) \to \mathcal{B}(\mathcal{K}_A)$ such that $\pi_A(g_x) = \widetilde{E}_{2,x} - \widetilde{E}_{1,x}$, where g_x is the generator of the x-th copy of $\mathbb{Z}_2$ in $*_3\mathbb{Z}_2$. Similarly, there is a unital $*$-homomorphism $\pi_B : C^*(*_3\mathbb{Z}_2) \to \mathcal{B}(\mathcal{K}_B)$ such that $\pi_B(g_x) = \widetilde{F}_{2,y} - \widetilde{F}_{1,y}$.

Set $\eta_i = Vf_i$ for $i = 1, 2, 3$, so that $\{\eta_1, \eta_2, \eta_3\}$ is orthonormal. Considering the $(3, 1)$-entry of equation (6.1.2), we have

$$\langle(\pi_A(g_3) \otimes \pi_B(g_3))\eta_1, \eta_3\rangle = 1. \tag{6.1.10}$$

Using the Cauchy-Schwarz inequality on equation (6.1.10), we have

$$\pi_A(g_3) \otimes \pi_B(g_3)\eta_1 = \eta_3. \tag{6.1.11}$$

Considering the $(1, 3)$ and $(2, 3)$ entries of equation (6.1.1), we see that

$$\langle\pi_A(g_2) \otimes \pi_B(g_2)\eta_3, \eta_1\rangle = \langle\pi_A(g_2) \otimes \pi_B(g_2)\eta_3, \eta_2\rangle = \frac{1}{\sqrt{2}}. \tag{6.1.12}$$

Now, let $T_1 = \pi_A(g_1) \otimes I$ and $T_2 = I \otimes \pi_B(g_1)$, and set $U_1 = \pi_A(g_2)\pi_A(g_3) = \pi_A(g_2g_3)$ and $U_2 = \pi_B(g_2)\pi_B(g_3) = \pi_B(g_2g_3)$. Combining equations (6.1.11) and (6.1.12), it follows that

$$\langle(U_1 \otimes U_2)\eta_1, \eta_2\rangle = \langle(U_1 \otimes U_2)\eta_1, \eta_1\rangle = \frac{1}{\sqrt{2}}.$$

Considering the $(1, 1)$ and $(2, 2)$ entries of equation (6.1.3), we also have

$$\langle(T_1 \otimes I)\eta_1, \eta_1\rangle = \langle(I \otimes T_2)\eta_1, \eta_1\rangle = 1$$

and

$$\langle(T_1 \otimes I)\eta_2, \eta_2\rangle = \langle(I \otimes T_2)\eta_2, \eta_2\rangle = -1.$$

Therefore, the unitaries T_1, T_2, U_1, U_2 and the unit vectors η_1, η_2 satisfy equations (6.0.5)–(6.0.7), contradicting Lemma 5.2.3. $\square$

6.2 Two inputs and three outputs

In this section, we will show that $C_q^{(n)}(2, 3) \neq C_{qs}^{(n)}(2, 3)$ for some $n \in \{2, 3, 4\}$. Since elements of $C_{qs}^{(n)}(2, 3)$ arise from tensor products of representations of $C^*(\mathbb{Z}_3 * \mathbb{Z}_3)$, we aim to transform the representation of $C^*(\mathbb{Z}_2 * \mathbb{Z})$ into some representation of $C^*(\mathbb{Z}_3 * \mathbb{Z}_3)$. However, we obtain a simpler group embedding (and hence a smaller number of required unit vectors) by first considering the embedding of $\mathbb{Z}_2 * \mathbb{Z}$ into $\mathbb{Z}_2 * \mathbb{Z}_3$ from Proposition 1.4.5. If g is the generator of $\mathbb{Z}_2$, h is a generator of $\mathbb{Z}_3$ and u is a generator of $\mathbb{Z}$, then we will use the embedding of $\mathbb{Z}_2 * \mathbb{Z}$ into $\mathbb{Z}_2 * \mathbb{Z}_3$ given by

$$g \mapsto g, \;\; u \mapsto hgh.$$

Using this embedding, we can translate the unitaries T and U from equation (6.0.3) and the unit vectors ζ_1 and ζ_2 from equation (6.0.4) to tensor product representations of $C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ that cannot be witnessed by tensor products of finite-dimensional representations. In this case, we only need to specify certain equations governing the representations and the unit vectors using words of length at most three.

Lemma 6.2.1. *There is an infinite-dimensional Hilbert space $\mathcal{H}$, a unital $*$-homomorphism $\sigma : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{H})$, and unit vectors $\zeta_1, \zeta_2 \in \mathcal{H} \otimes \mathcal{H}$ such that*

$$\langle (\sigma(hgh) \otimes \sigma(hgh))\zeta_1, \zeta_1 \rangle = \langle (\sigma(hgh) \otimes \sigma(hgh))\zeta_1, \zeta_2 \rangle = \frac{1}{\sqrt{2}}, \tag{6.2.1}$$

$$\langle (\sigma(g) \otimes I)\zeta_1, \zeta_1 \rangle = \langle (I \otimes \sigma(g))\zeta_1, \zeta_1 \rangle = 1, \tag{6.2.2}$$

$$\langle (\sigma(g) \otimes I)\zeta_2, \zeta_2 \rangle = \langle (I \otimes \sigma(g))\zeta_2, \zeta_2 \rangle = -1. \tag{6.2.3}$$

Moreover, these equations cannot be witnessed by a tensor product of finite-dimensional representations of $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$.

Proof. Let $\mathcal{S} = \text{span}\,\{1, g, hgh, h^2gh^2\} \subseteq C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$. Since $g^* = g$ and $(hgh)^* = h^*gh^* = h^2gh^2$, $\mathcal{S}$ is an operator system. Applying Proposition 1.4.1, we see that the group embedding $\mathbb{Z}_2 * \mathbb{Z} \hookrightarrow \mathbb{Z}_2 * \mathbb{Z}_3$ from Proposition 1.4.5 induces an injective $*$-homomorphism $C^*(\mathbb{Z}_2 * \mathbb{Z}) \hookrightarrow C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$, with a completely positive expectation onto the range. Restricting to $\mathcal{S}$, we obtain a complete order isomorphism of the operator system $\mathcal{P} = \text{span}\,\{1, g, u, u^*\} \subseteq C^*(\mathbb{Z}_2 * \mathbb{Z})$ onto $\mathcal{S}$ via $g \mapsto g$ and $u \mapsto hgh$. Let $\pi : C^*(\mathbb{Z}_2 * \mathbb{Z}) \to \mathcal{B}(\ell^2(\mathbb{Z}))$ be the unital $*$-homomorphism given by $\pi(g) = T$ and $\pi(u) = U$, where T, U are the operators in equation (6.0.3). Since $\mathcal{S} \simeq \mathcal{P}$, the restriction of π to $\mathcal{S}$ gives a unital completely positive map $\psi : \mathcal{S} \to \mathcal{B}(\ell^2(\mathbb{Z}))$. By Arveson's extension theorem [2], we may extend ψ to a unital completely positive map $\varphi : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\ell^2(\mathbb{Z}))$. Using Stinespring's dilation theorem [61], there is a Hilbert space $\mathcal{H}$, a unital $*$-homomorphism $\sigma : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{H})$ and an isometry $V : \ell^2(\mathbb{Z}) \to \mathcal{H}$ such that $\varphi(\cdot) = V^*\sigma(\cdot)V$. Since V is an isometry, we identify $\ell^2(\mathbb{Z})$ with $V\ell^2(\mathbb{Z})$ and write $\mathcal{H} = \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z})^\perp$.

Since $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \subseteq \mathcal{H} \otimes \mathcal{H}$, we may identify $\zeta_1 = \sum_{j<0} (\sqrt{2}) e_j \otimes e_j$ and $\zeta_2 = e_0 \otimes e_0 \in \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ as unit vectors in $\mathcal{H} \otimes \mathcal{H}$. It is not hard to check that equations (6.2.1)–(6.2.3) are satisfied.

Suppose that there are unital $*$-homomorphisms $\pi_A : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{H}_A)$ and $\pi_B : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{H}_B)$, along with unit vectors $\zeta_1, \zeta_2 \in \mathcal{H}_A \otimes \mathcal{H}_B$ satisfying equations (6.2.1)–(6.2.3), where $\mathcal{H}_A$ and $\mathcal{H}_B$ are finite-dimensional. Then setting $T_1 = \pi_A(g)$, $T_2 = \pi_B(g)$, $U_1 = \pi_A(hgh)$ and $U_2 = \pi_B(hgh)$, we would yield equations (6.0.5)–(6.0.7) on a tensor product of finite-dimensional Hilbert spaces, contradicting Lemma 5.2.3. $\square$

We are now in a position to show that $C_q^{(4)}(2,3) \neq C_{qs}^{(4)}(2,3)$. For convenience, for $n \geq 2$, we will let $Q_n : \mathbb{C}^2 \to \mathbb{C}^n$ be the isometry sending $\mathbb{C}^2$ to the first two coordinates of $\mathbb{C}^n$. We also let $\omega = \exp\left(\frac{2\pi i}{3}\right)$.

Theorem 6.2.2. *There exist $n \in \{2,3,4\}$, contractions A and B in M_n, and an element $P = P(a,b|x,y) \in C_{qs}^{(n)}(2,3)$ such that*

$$P(2,2|1,1) - P(1,2|2,2) - P(2,1|2,2) + P(1,1|2,2) = A, \tag{6.2.4}$$

$$P(3,b|1,y) = P(a,3|x,1) = 0, \forall a,b,x,y \tag{6.2.5}$$

$$\sum_{a,b=1}^{3} \omega^{a+b} P(a,b|2,2) = B, \tag{6.2.6}$$

$$Q_n^*(P_A(2|1) - P_A(1|1))Q_n = Q_n^*(P_B(2|1) - P_B(1|1))Q_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.2.7}$$

$$\sum_{i=1}^{n} |B_{i1}|^2 = 1, \tag{6.2.8}$$

$$\sum_{i=1}^{n} |(AB)_{i1}|^2 = 1, \tag{6.2.9}$$

$$(BAB)_{1,1} = (BAB)_{2,1} = \frac{1}{\sqrt{2}}. \tag{6.2.10}$$

Moreover, for these contractions A and B, if $\widetilde{P} \in C_{qs}^{(n)}(2,3)$ satisfies equations (6.2.4)–(6.2.10), then $\widetilde{P} \notin C_q^{(n)}(2,3)$.

Proof. Let $\sigma : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{H})$ and $\zeta_1, \zeta_2 \in \mathcal{H} \otimes \mathcal{H}$ be as in Lemma 6.2.1. Then σ_1, σ_2 and ζ_1, ζ_2 satisfy equations (6.2.1)–(6.2.3). We define intermediate unit vectors

$$\xi_3 = (\sigma(h) \otimes \sigma(h))\zeta_1, \tag{6.2.11}$$

$$\xi_4 = (\sigma(g) \otimes \sigma(g))\xi_3 = (\sigma(gh) \otimes \sigma(gh))\zeta_1. \tag{6.2.12}$$

Although $\{\zeta_1, \zeta_2\}$ is orthonormal, the set $\{\zeta_1, \zeta_2, \xi_3, \xi_4\}$ may not be orthonormal. Applying the Gram-Schmidt orthonormalization process, we obtain an orthonormal basis $\zeta_1, ..., \zeta_n$ for the subspace $\mathcal{M} = \text{span}\,\{\zeta_1, \zeta_2, \xi_3, \xi_4\}$ of $\mathcal{H} \otimes \mathcal{H}$, where $n = \dim(\mathcal{M}) \in \{2,3,4\}$. Define $W : \mathbb{C}^n \to \mathcal{H} \otimes \mathcal{H}$ by $We_i = \zeta_i$ for $1 \leq i \leq n$; then W is an isometry. Since $\sigma(g)$ is a self-adjoint unitary, we may write $\sigma(g) = E_{2,1} - E_{1,1}$ for a PVM $\{E_{1,1}, E_{2,1}\}$ on $\mathcal{H}$. We extend this PVM to have three outputs by setting $E_{3,1} = 0$. Since $\sigma(h)$ is an order three unitary, there is a PVM $\{E_{1,2}, E_{2,2}, E_{3,2}\}$ on $\mathcal{H}$ such that $\sigma(h) = \sum_{b=1}^{3} \omega^b E_{b,2}$. Define $F_{b,y} = E_{b,y}$ for all $y = 1, 2$ and $b = 1, 2, 3$, and set

$$P(a, b|x, y) = W^*(E_{a,x} \otimes F_{b,y})W. \tag{6.2.13}$$

Then $P = (P(a, b|x, y))_{a,b,x,y}$ defines an element of $C_{qs}^{(n)}(2, 3)$. Putting together equations (6.2.11) and (6.2.12), we recover the equation

$$(\sigma(h) \otimes \sigma(h))\xi_4 = (\sigma(hgh) \otimes \sigma(hgh))\zeta_1 = \frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2). \tag{6.2.14}$$

Define elements A, B, C, D of M_n by

$$A = W^*(\sigma_1(g) \otimes \sigma_2(g))W, \quad B = W^*(\sigma_1(h) \otimes \sigma_2(h))W \tag{6.2.15}$$

$$C = W^*(\sigma_1(g) \otimes I)W, \qquad D = W^*(I \otimes \sigma_2(g))W. \tag{6.2.16}$$

Then A, B, C and D are contractions. Moreover, since

$$\sigma(g) \otimes \sigma(g) = E_{2,1} \otimes F_{2,1} - E_{1,1} \otimes F_{2,1} - E_{2,1} \otimes F_{1,1} + E_{1,1} \otimes F_{1,1},$$

the correlation P satisfies equation (6.2.4). Similarly, considering $\sigma(h) \otimes \sigma(h)$, P must satisfy equation (6.2.6). Equation (6.2.5) follows since $E_{3,1} = F_{3,1} = 0$. By the choice of the unitary $\sigma(g)$ and the unit vectors ζ_1 and ζ_2, equations (6.2.2) and (6.2.3) show that

$$Q_n^* C Q_n = Q_n^* D Q_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2.17}$$

Since $C = P_A(2|1) - P_A(1|1)$ and $D = P_B(2|1) - P_B(1|1)$, we obtain equation (6.2.7). Combining equations (6.2.11), (6.2.15), for each $1 \leq i \leq 4$,

$$\langle \xi_3, \zeta_i \rangle = \langle (\sigma(h) \otimes \sigma(h))\zeta_1, \zeta_i \rangle = B_{i1}. \tag{6.2.18}$$

Since $\xi_3 \in \mathcal{M} = \text{span}\,\{\zeta_1, ..., \zeta_n\}$, it follows that

$$\xi_3 = \sum_{j=1}^{n} B_{j1}\zeta_j. \tag{6.2.19}$$

Moreover, since $\{\zeta_1, ..., \zeta_n\}$ is an orthonormal basis for $\mathcal{M}$ and ξ_3 is a unit vector, we have

$$1 = \|\xi_3\|^2 = \sum_{j=1}^{n} |B_{j1}|^2. \tag{6.2.20}$$

Thus, B satisfies equation (6.2.8). Using equation (6.2.12) and applying $\sigma(g) \otimes \sigma(g)$ to equation (6.2.19), we obtain

$$\langle \xi_4, \zeta_i \rangle = \left\langle (\sigma(g) \otimes \sigma(g)) \left(\sum_{j=1}^{n} B_{j1} \right) \zeta_j, \zeta_i \right\rangle \tag{6.2.21}$$

$$= \sum_{j=1}^{n} B_{j1} \langle (\sigma(g) \otimes \sigma(g)) \zeta_j, \zeta_i \rangle \tag{6.2.22}$$

$$= \sum_{j=1}^{n} A_{ij} B_{j1} = (AB)_{i1}, \tag{6.2.23}$$

where $(AB)_{i1}$ denotes the $(i,1)$-entry of AB. Using the fact that $\xi_4 \in \mathcal{M}$, we obtain

$$\xi_4 = \sum_{i=1}^{n} (AB)_{i1} \zeta_j. \tag{6.2.24}$$

Since ξ_4 is a unit vector, the first column of AB has norm 1, which yields equation (6.2.9). An analogous argument with equation (6.2.14) demonstrates that

$$\frac{1}{\sqrt{2}}(\zeta_1 + \zeta_2) = \sum_{i=1}^{n} (BAB)_{i,1} \zeta_i, \tag{6.2.25}$$

which forces equation (6.2.10) to be satisfied. We conclude that P, A and B satisfy all of equations (6.2.4)–(6.2.10).

Now, suppose for a contradiction that there is $\widetilde{P}$ in $C_q^{(n)}(2,3)$ satisfying equations (6.2.4)–(6.2.10). Then there are finite-dimensional Hilbert spaces $\mathcal{K}_A$ and $\mathcal{K}_B$, PVMs $\{\widetilde{E}_{a,x}\}_{a=1}^{3}$ on $\mathcal{K}_A$ for $x = 1, 2$, and PVMs $\{\widetilde{F}_{b,y}\}_{b=1}^{3}$ on $\mathcal{K}_B$ for $y = 1, 2$, along with an isometry $V : \mathbb{C}^n \to \mathcal{K}_A \otimes \mathcal{K}_B$ such that

$$\widetilde{P} = (V^*(\widetilde{E}_{a,x} \otimes \widetilde{F}_{b,y})V)_{a,b,x,y} \in C_q^{(n)}(2,3).$$

By equation (6.2.5), $\widetilde{P}(3,b|1,y) = \widetilde{P}(a,3|x,1) = 0$ for all a,b,x,y. By replacing $\widetilde{E}_{2,1}$ with $\widetilde{E}_{2,1} + \widetilde{E}_{3,1}$ if necessary, we may assume without loss of generality that $\widetilde{E}_{3,1} = 0$, and that $\{\widetilde{E}_{1,1}, \widetilde{E}_{2,1}\}$ is a PVM. Similarly, we may assume that $\widetilde{F}_{3,1} = 0$, and that $\{\widetilde{F}_{1,1}, \widetilde{F}_{2,1}\}$ is a PVM. Since $\widetilde{E}_{2,1} - \widetilde{E}_{1,1}$ is a self-adjoint unitary and $\sum_{a=1}^{3} \omega^a \widetilde{E}_{a,2}$ is an order three unitary, the map $\gamma_A : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{K}_A)$ given by $\gamma_A(g) = \widetilde{E}_{2,1} - \widetilde{E}_{1,1}$ and $\gamma_A(h) = \sum_{a=1}^{3} \omega^a \widetilde{E}_{a,2}$

extends to a unital $*$-homomorphism. Similarly, there is a unital $*$-homomorphism $\gamma_B : C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \to \mathcal{B}(\mathcal{K}_B)$ such that $\gamma_B(g) = \widetilde{F}_{2,1} - \widetilde{F}_{1,1}$ and $\gamma_B(h) = \sum_{b=1}^{3} \omega^b \widetilde{F}_{b,2}$. Since $\widetilde{P}$ satisfies equations (6.2.4), (6.2.6) and (6.2.7), it follows that

$$V^*(\gamma_A(g) \otimes \gamma_B(g))V = A, \;\; V^*(\gamma_A(h) \otimes \gamma_B(h))V = B, \text{ and} \tag{6.2.26}$$

$$Q_n^* V^*(I_{\mathcal{K}_A} \otimes \gamma_B(g))VQ_n = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Q_n^* V^*(\gamma_A(g) \otimes I_{\mathcal{K}_B})VQ_n. \tag{6.2.27}$$

For $1 \leq i \leq n$, define $\eta_i = Ve_i$. Then $\{\eta_1, ..., \eta_n\}$ is an orthonormal set. We define vectors

$$\chi_3 = \sum_{i=1}^{n} B_{i1}\eta_i, \text{ and} \tag{6.2.28}$$

$$\chi_4 = \sum_{i=1}^{n} (AB)_{i1}\eta_i. \tag{6.2.29}$$

These are unit vectors by equations (6.2.8) and (6.2.9). Applying the Cauchy-Schwarz inequality and noting that $B_{i1} = \langle (\gamma_A(h) \otimes \gamma_B(h))\eta_1, \eta_i \rangle$, it readily follows that

$$(\gamma_A(h) \otimes \gamma_B(h))\eta_1 = \chi_3, \text{ and} \tag{6.2.30}$$

$$(\gamma_A(g) \otimes \gamma_A(g))\chi_3 = \chi_4. \tag{6.2.31}$$

Using equation (6.2.10) and applying Cauchy-Schwarz again, it follows that

$$(\gamma_A(hgh) \otimes \gamma_B(hgh))\eta_1 = (\gamma_A(h) \otimes \gamma_B(h))\chi_4 = \frac{1}{\sqrt{2}}(\eta_1 + \eta_2). \tag{6.2.32}$$

A similar argument using equation (6.2.27) demonstrates that

$$(\gamma_A(g) \otimes I)\eta_1 = (I \otimes \gamma_B(g))\eta_1 = \eta_1, \tag{6.2.33}$$

$$(\gamma_A(g) \otimes I)\eta_2 = (I \otimes \gamma_B(g))\eta_2 = -\eta_2. \tag{6.2.34}$$

Thus, combining equations (6.2.32)–(6.2.34), we can realize equations (6.2.1)–(6.2.3) in a finite-dimensional tensor product setting, which contradicts Lemma 6.2.1. Therefore, we obtain the separation $C_q^{(n)}(2,3) \neq C_{qs}^{(n)}(2,3)$, as desired. □

Combining Theorems 6.1.1 and 6.2.2 and applying Proposition 1.8.8 shows that, for any (m,k) with $m,k \geq 2$ and $(m,k) \neq (2,2)$, we have $C_q^{(n)}(m,k) \neq C_{qs}^{(n)}(m,k)$ for some matrix level n, with $n \leq 4$. This result is optimal with respect to the input and output sets. Indeed, if $m = k = 2$, then we have

$$C_q^{(n)}(2,2) = C_{qs}^{(n)}(2,2) = C_{qa}^{(n)}(2,2),$$

since the underlying group, $\mathbb{Z}_2 * \mathbb{Z}_2$, is amenable and has the property that every irreducible representation is at most 2-dimensional. On the other hand, while it is still unknown whether $C_q(3,2) \neq C_{qs}(3,2)$ or $C_q(2,3) \neq C_{qs}(2,3)$, Theorem 6.0.11 provides some partial evidence that these separations may possibly hold.

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